Replicating Exemplary Practices in Mathematics Education among APEC Economies

APEC Human Resource Development Working Group

July 2010

APEC Project HRD 01/2009 A

Prepared By:
Akihiko Takahashi
DePaul University
atakahas@depaul.edu

Brian Fu
U.S. Department of Education
brian.fu@ed.gov

Produced for
Asia-Pacific Economic Cooperation Secretariat
35 Heng Mui Keng Terrace Singapore 119616
Tel: (65) 6891-9600 Fax: (65) 6891-9690
Email: info@apec.org Website: www.apec.org

APEC#210-HR-01.4

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Preface

Understanding mathematical principles and procedures is essential in becoming a citizen of the data-driven and technological world of the 21st century, no matter what industry one is in. Mathematics education is indeed a key for human resources development and global competitiveness.

The Education Ministers of the Asia-Pacific Economic Cooperation (APEC) identified mathematics and science education as a priority in their most recent ministerial meeting in 2008. There, they released a strategic action plan and recommendations for the APEC Education Network. These recommendations recognize not only the need for high quality standards and assessments for mathematics education, but also the need for teachers with strong knowledge and expertise in providing high-quality learning opportunities for their students.

The APEC project, “21st Century Mathematics and Science Education for All in the APEC Region: Strengthening Developing Economies and Gender Equity Through Standards, Assessments, and Teachers”, led by the U.S. and Thailand intends to make available promising practices and ideas from research on improving mathematics teaching and learning to all the APEC economies[1].

The main goals of this project are to: share exemplary practices in mathematics education from around the APEC region, and develop technical assistance from these promising practices to help APEC developing economies effectively replicate these practices based on their individual contexts.

As a part of this project the APEC Conference on Replicating Exemplary Practices in Mathematics Education was held from March 8 to March 12, 2010, with March 8 designated as a special one-day preconference event focusing on gender equity in mathematics and science education, at the International School of Tourism, Suratthani Rajabhat University, Samui Island. The conference was organized and led by the U.S. Agency for International Development, U.S. Department of Education and the Ministry of Education, Thailand.

This summary addresses: (1) What was discussed during the conference (2) The planned next steps in using these resources to provide high-quality learning experiences for teachers and students, and to establish a strong foundation for teaching mathematics and science for future generations.

The conference report

The ultimate goal of the conference was to develop a series of resources, recommendations, and action plans for the APEC Education Network mathematics project participants based on the presentations and discussions held during the conference. In order to address issues and concerns for improving mathematics teaching and learning, the project overseers and the conference chairs identified five major topics in mathematics education. Based on these topics, the 4-day conference was organized around five major topics and was designed for intensive discussion on each topic among not only the speakers and discussants but also including the active participants who were nominated by the each APEC member economy. The following

1 This is project was approved by the APEC, Human Resources Development Working Group [HRD 01 2009A]

are the five plenary sessions based on the five major topics that were identified by the conference organizers:

1. Standards plenary session
2. Curriculum plenary session
3. Teacher plenary session
4. Assessment plenary session
5. Interventions plenary session

Each plenary session began with presentations from selected monograph writers followed by break-out sessions, which provided the participants an opportunity to engage in discussion with the presenters and other active participants. Next, the whole group reconvened for panel discussions, which included all presenters of the plenary session and the discussants who represented the audience members of the resources from this project. 17 distinguished presenters and 5 discussants from 12 economies were invited and participated in the conference.

Standards Plenary Session

Based on the three session papers, the discussants and the active participants discussed what mathematical knowledge and skills are expected, and how these knowledge and skills should be introduced for the future generations. Ginsburg and Leinwand shared the results of the study comparing the standards from high-performing APEC economies, such as Hong Kong, China; Korea; and Singapore. Usiskin shared his ideas on the learning progression in grade 7-12 mathematics content. Wang shared the results from his study on using a new technology based curriculum.

The discussant of the session, Lim, highlighted some of the ideas and proposals that were discussed during the session:

- Gather, translate (into English) and post additional sets of standards from economies that are not currently available.
- Develop a set of “Guidelines for Conducting International Benchmarking Standards” including specific techniques or approaches for adapting a set of standards

Curriculum Plenary Session

The three session papers delved into the processes and principles for the development of mathematics textbooks based on standards and gave innovative ideas and examples on how textbook publishers, curriculum developers, and authors can ensure a more effective way for implementing them into the curricula. The first paper written by Shimizu and Watanabe discussed how mathematics textbooks were produced in Japan. The paper highlighted the processes involved in textbook writing and the roles played by the Ministry of Education and commercial publishers. The second paper, written by Lianghuo discussed the relevant processes necessary for developing mathematics textbooks based on 6 principles: Curriculum, Discipline, Pedagogy, Technology, Context and Presentation. The third presentation, by Ginsburg gave ideas on the principles and processes of the best way online resources can be published while keeping curriculum principles in mind.

The discussants of the session, Kaur and Soh, mentioned that it would be beneficial for many APEC economies to have guidance or a manual for textbook writers and curriculum developers about the exemplary process of developing textbooks, reviewing the quality of textbooks, and the adaptation of textbooks. In order to do so, an in-depth study on comparing the practices among some of the high-performing economies is necessary. It is also suggested that seeking possibilities of publishing online curriculum materials as OpenCourseWare would be greatly beneficial for many APEC economies. Some of the participants agreed that they were going to make some sample materials available on the APEC KnowledgeBank Wiki site in the near future.

Teacher Plenary Session

The four papers presented at the teacher plenary session touched on some of the most important areas of mathematics teacher education: professional development, performance measures and reward systems, and a particular professional development approach to enhancing teachers' capacities in the classroom – Lesson Study. Akihiko Takahashi presented about a framework of professional development for teachers to grow professionally throughout their careers. Lee's paper describes the Singapore government's role in helping develop excellent teachers. Isoda and Inprasitha both consider the process and the adaptation of Lesson Study as a specific approach to developing excellent teachers.

The discussant of the session, Vistro-Yu, reported that the break-out session discussion focused on the mathematics preparation needed by teachers, professional development for teacher educators, the quality of beginning teachers, Singapore's system and methods for maintaining an excellent teaching force, and details of Thailand's experience with Lesson Study using the Open Approach as opposed to the Top-Down approach for mathematics teachers' professional development.

Although the participants agreed that professional development of teachers is a key for improving mathematics teaching and learning, the question of what kind of professional development works best for teachers still remains. For the teachers themselves, a content-based professional development is needed to address their weak content knowledge. However, to address the poor quality of teaching, pedagogical content knowledge that is deeply related to content must be their focus. In order to provide high-quality professional development for teachers, the participants agreed that the quality of professional development for professional developers (or PDPDers) should also be emphasized. Whatever framework is developed for the professional development of teacher trainers and PDers could be disseminated through Open Educational Resources (OER). Likewise, PDPDers could deliver materials using online OER courses. Participants also identified a need to compile resources, address practical issues of delivery, and provide various models of professional development. As for resources including textbooks, a practical issue that needs serious consideration is language. Works in English have to be translated to the language of the economy and vice versa.

Assessment Plenary Session

The three papers presented at the assessment plenary session described some of the key aspects of assessments. Cheng reported the current issues and trends in high school competency exams in Chinese Hong Kong, China. Stevens described New Zealand's example of the systematic use of formative assessment to improve student achievement. Anderson shared the approach of using formative assessment used in many international schools.

The discussant of the session, Suwaryani, reported that the participants agreed that both formative and summative assessments were important for ensuring the quality of the student learning. At the same time, some teachers may not able to see formative and summative assessment as an integral part of attempts to improve student learning, and may not know how to follow up on the results of the assessment.

Since teachers in developing economies are often required to teach students with diverse ability, including multi-grade teaching, knowledge and skill for using formative assessment to improve students learning is essential. Given these circumstances, having sets of assessment tasks and a guide for using assessment effectively in the APEC KnowledgeBank Wiki would be an ideal step following this conference.

Intervention Plenary Session

Three papers were presented at the intervention plenary session. Each paper described a unique intervention program for students with special needs. Cobb and Crombie shared ideas from the Algebra Project. The authors describe the Algebra Project, which is funded by Bob Moses, as a direct descendent of the community-organizing tradition of America's Civil Rights Movement. The project is designed to accelerate the mathematical learning of students who are under-performing in mathematics.

Gould reported the results of the, "Taking Off With Numeracy" (TOWN), which builds upon the work of the past ten years in the "Counting On" program and uses essentially the same learning framework employed in "Math Recovery" and "Count Me In Too". The program was designed to address the persistence of highly inefficient methods of calculating, which operates as both a whole class program and a within class intervention.

Vui gave some evidence showing the advantages of using multiple dynamic representations in promoting the exploration of mathematical ideas in mathematically gifted students. He argued that these students may need more help from outside schools, while on the other hand lower math achievers may need more assistance from inside schools.

The discussant of the session, Bao, pointed out some of the important considerations needed for establishing effective intervention programs by summarizing the discussion at the conference:

- Motivation principle is a key concept. Since only students can prevent themselves from being low achievers, establishing community organizations to encourage students, reducing class size to provide more attention to individual students, and providing for more hands on activities are important considerations for helping them catch up.
- Need to design intervention programs according to needs and target the greatest impact (i.e. identify and address the tipping point).
- Need to understand strategies to transfer promising practices including evolving and modifying practices to be implemented effectively in different economies.

Informing Grades 1–6 Mathematics Standards Development: What Can Be Learned From High-Performing with Korea, Singapore, and Hong Kong, China?

Alan Ginsburg, U.S. Department of Education
Steven Leinwand, American Institutes for Research
Katie Decker, University of Iowa

The United States is embarking on a historic policy reversal as it moves toward developing common education standards in reading and mathematics. Supporting this movement is the U.S. Department of Education's $4.3 billion Race-to-the-Top (RttT) competition under the American Recovery and Reinvestment Act (ARRA) and the Common Core State Standards Initiative sponsored by the National Governors Association and the Council of Chief State School Officers. To inform these efforts, this report examines what it means to internationally benchmark mathematics standards for grades 1–6 against the composite standards of three high-mathematics performing Asian economies: Hong Kong, China; Korea; and Singapore.

The U.S. common standards movement offers a unique opportunity to address a well-documented weakness found in many State mathematics standards: many topics are taught in a single grade and many topics are repeated over several grades. This topic spread has led to the well-known characterization of U.S. elementary mathematics curriculum expectations as "a mile wide and an inch deep" (Schmidt, Houang, & Cogan, 2002).

The move to common internationally benchmarked standards offers an opportunity to model U.S. standards off of those of high performing economies such as Singapore, which offer a more coherent and focused set of expectations. The composite Hong Kong, China; Korea; and Singapore standards developed in this report present one effort to internationally benchmark grades 1–6 mathematics standards against high-performing nations.

Methodology

The Hong Kong, China; Korean; and Singapore standards were chosen for international benchmarking because of their high performance on the Trends in International Mathematics and Science Study (TIMSS) assessments and the availability of these standards in English on the Asian Pacific Economic Cooperation (APEC) web site. Because of the concern over lack of rich mathematical content progression with many U.S. state standards, our particular focus in developing a composite set of standards is on learning progressions – how systematically the mathematical content progresses across the grades within a broad mathematical topic. The mathematics standards for the three economies are available at http://hrd.apec.org, the website of the Human Resource Development (HRD) working group of the Asian Pacific Economic Cooperation (APEC), an organization composed of 21 economies bordering the Pacific Ocean.

The development of the composite Korea, Singapore and Hong Kong, China mathematics standards was conducted in three steps: (1) identify the *core mathematics topics* taught within each strand across economies; (2) identify each economy's grade-by-grade *sequence of* *mathematical competencies* for each core topic; and (3) create *composite standards* for each core topic by drawing from the learning progressions in the standards of each of the three economies.

Identify the core mathematics topics for each strand. These topics are generally apparent from the topic structure of the standards. The Korean standards present a summary table of "Content Organization" that identifies the major topics. The Singapore and Hong Kong, China standards explicitly identify the topics and subtopics associated with each strand by grade. Table 2 shows the core mathematics topics identified within each strand based on the three sets of standards.

Table 2. Core Mathematics Topics: Grades 1–6

Numbers	Measurement	Geometry	Data/Probability	Patterns/Algebra
• Whole numbers • Addition/ subtraction • Multiplication/ division • Fractions • Decimals • Ratios • Percents	• Linear measurement • Perimeter / area • Volume • Nongeometric measurement – *Time (clock)* – *Time (calendar)* – *Money* – *Weight*	• 2-D shapes • 3-D shapes • Lines • Angles	• Classification of objects by attributes into groups • Pictograms • Bar graphs • Tables • Line graphs • Averages • Pie charts	• Symbols • Equations

For each core topic, identify the learning progressions across grades for that topic from each standard. This underlying concept of learning progressions has been described by the National Assessment of Educational Progress (NAEP), as follows:

> A learning progression is a sequence of successively more complex ways of reasoning about a set of ideas. … In other words, the progression from novice learner to competent learner to expert begins with the acquisition of relevant experiences, principles, concepts, facts, and skills and moves to the accumulation and organization of knowledge in a specific domain and finally to expertise *after extensive experience and practice.* (National Assessment Governing Board, 2008)

The learning progressions for each economy's standards by core topic were constructed by pulling out the content for that core topic grade by grade. For example, the content of the linear measurement topic was identified for each grade by economy.

Create the composite standards. First, the grades covered for each topic were compared across the three sets of standards to ensure similarity. For most topics, the three sets of standards cover the topics over similar grades, although exceptions often occurred at the end points. An illustration of a typical grade pattern is the "measurement of length" presented in Table 3. All three sets of national standards cover length measures in grades 1–3 and Singapore covers measurement of length in grade 5 as a learning objective connected with showing an application of decimals. An outlier grade (e.g., grade 5 Singapore for measurement of length) was included in the composite standard if it was judged to add important content to the learning progression.

Table 3. Measurement of Length by Grade: Hong Kong (H), Korea (K), and Singapore (S)

	Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
Length	H, K, S	H, K, S	H, K, S		S	

The composite standards for each core mathematics topic were then created by consolidating the competencies from the three sets of standards that describe what students should know and be able to do across the grades. The composite standards represent a judgment designed to present the essential learning competencies from the three sets of standards that most clearly indicate the learning progression for each topic over the grades. The following rules were employed in creating the composite standards:

- If core topic contents were similar for the three standards grade by grade, the content of the standard that was judged to offer the clearest learning progression was chosen.
- If core topic contents were similar overall across standards but differed in some respects grade by grade, the composite standard reflected a judgment as to which standard offered the most in-depth or clearest learning progression.
- If a core topic contents differed on some competency that was not in the other standards, that competency was included only if it added to and was consistent with the learning progression.

Table 4 shows a sample learning progression for the topic of length within the measurement strand, with some key features:

- The learning progression contains two broad sequences or subtopics, one for concept of length and a second for tools/measuring length.
- The concept of length sequence introduces the concept in grade 1 along with the centimeter unit and proceeds in later grades to present different sizes of measuring units and decimals.
- The tools/measuring length sequence presents similar skills at each of grades 1, 2, and 3, but with a clear progression in the selection of the measurement units used.

Note that the composite standards represent the progression of core learning objectives for a topic and may omit some of the details associated with the standards for a particular topic from any of the three economies.

Table 4. Composite Learning Progression for Length Within Measurement

Measurement Topic	Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
Concept of length	• Understand concepts of length and distance • Understand long, longer, longest, short, shorter, shortest • Understand centimeter	• Understand that a meter is greater than a centimeter	• Understand that a kilometer is greater than a meter and that a millimeter is smaller than a centimeter • Understand that—Convert compound units to a smaller or a larger unit		• Convert from a smaller unit to a larger unit and vice versa in decimal form	
Tools/measuring length	• Measure and compare lengths of objects and distance with centimeters • Estimate lengths and distances • Measure length with appropriate tools	• Measure and compare length and distance in meters and centimeters • Estimate lengths and distances • Measure length with appropriate tools	• Measure and compare length and distance in kilometers and millimeters • Estimate lengths and distances • Measure length with appropriate tools			

Sample Learning Progressions

Whole numbers. Counting is one of the first skills that children learn in mathematics. Along with rote, sequential counting, children learn to count and compare the number of objects in sets. Counting progresses with larger numbers over the grades as place value concepts are learned. With larger numbers, children apply their place value understanding to comparing, ordering, and rounding numbers. Numbers are further differentiated into odd and even groups.

Table 5 shows the composite set of standards for whole numbers.

- Grade 1 addresses whole numbers up to 100. Basic whole-number skills include counting the numbers of objects in a set which requires one-to-one correspondence between the objects and the number; comparing the size of sets; ordering numbers; and knowing that numbers show position (1st, 2nd). Place value is introduced to distinguish tens and ones, and the correspondence between numeral symbols and words is taught.
- The grade 2 learning progression extends recognizing and ordering whole numbers up to 1,000 and understanding place value of hundreds, tens, and ones and adding 10 and 100 to numbers mentally.
- Grade 3 focuses on numbers up to 10,000 with place value to thousands.
  At grade 4, numbers are up to 100,000, and rounding and approximation are introduced. Grade 5 explicitly treats understanding of large numbers up to a hundred million along with the concepts of approximation, estimation, and rounding.

Table 5. Composite Standards: Numbers—Whole Numbers

Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
Whole numbers to 100: • Count to tell the number of objects in a given set • Count forward and backward • Compare the number of objects in two or more sets • Use ordinal numbers (first, second, up to tenth) and symbols (1st, 2nd, 3rd, etc.) • Use number notation and place values (tens, ones) • Read and write numbers in numerals and in words • Compare and order numbers	Whole numbers to 1,000: • Count in tens and hundreds • Use number notation and place values (hundreds, tens, ones) • Read and write numbers in numerals and in words • Compare and order numbers	Whole numbers to 10,000: • Use number notation and place values (thousands, hundreds, tens, ones) • Read and write numbers in numerals and in words • Compare and order numbers • Understand odd and even numbers	Whole numbers to 100,000: • Use number notation and place values (ten thousands, thousands, hundreds, tens, ones) • Read and write numbers in numerals and in words • Compare and order numbers • Round numbers to the nearest 10 or 100	• Develop an understanding of large numbers • Develop the concept of approximation • Estimate the number of a large quantity of objects • Round large numbers in thousands, ten thousands, hundred thousands, millions, ten millions, hundred millions	

Addition and subtraction. The number standards present addition along with subtraction so that students understand the meanings and relationship. The standards require mastering adding and subtracting smaller numbers, thereby promoting automaticity. The learning progression builds up addition and subtraction skills by introducing successively larger numbers over grades. Word problems are also introduced early on as a way of promoting students' understanding of addition and subtraction concepts (Har & Hoe, 2007). Table 6 shows the composite set of standards for addition and subtraction.

- The concepts of addition and subtraction are introduced in grade 1. Addition and subtraction is initially constrained within 20 and includes learning all the different combinations of sums through 9 plus 9 and finding an unknown number within the combination. By the end of the year, addition and subtraction expand to sums and differences within 100 without regrouping and the introduction of 1-step word problems. Addition and subtraction facts are taught in all three economies using what Singapore calls "number bonds" and the United States calls "fact families" that relate, for example, 4 + 7, 7 + 4, 11 – 4, and 11 – 7 to promote understanding and reduce memory load.
- In grade 2, addition and subtraction are extended to numbers involving three digits, 2-step word problems, and mental calculation.

- In grade 3, addition and subtraction are extended to numbers up to four digits, and again there is a stress on word problems.

Table 6. Composite Standards: Numbers—Addition and Subtraction

Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
Addition and subtraction: • Understand situations for , and the meaning of, addition and subtraction • Use the addition symbol (+) or the subtraction symbol (−) • Compare two numbers within 20 to tell how much one number is greater (or smaller) than the other • Recognize the relationship between addition and subtraction • Build the addition bonds up to 9 + 9 • Solve 1-step word problems involving addition and subtraction within 20 • Add more than two 1-digit numbers • Add and subtract within 100 without regrouping involving – *a 2-digit number and ones* – *a 2-digit number and tens* – *two 2-digit numbers* • Use mental calculation for addition and subtraction – *within 20* – *involving a 2-digit number and ones without renaming* – *Involving a 2-digit number and tens*	Addition and subtraction of numbers up to three digits: • Solve up to 2-step word problems involving addition and subtraction • Use mental calculation for addition and subtraction involving – *a 3-digit number and ones* – *a 3-digit number and tens* – *a 3-digit number and hundreds*	Addition and subtraction of numbers up to four digits: • Use the terms “sum” and “difference” • Solve up to 2-step word problems involving addition and subtraction			

Perimeter and area. Recognizing that perimeter is a measure of length, perimeter and area are treated together because they both measure different attributes of two-dimensional shapes. The three Asian economies differ slightly in when students first encounter these concepts, with Singapore beginning in grade 3 and Korea and Hong Kong, China in grade 4. Our measurement composite elects to follow the Singapore approach beginning in grade 3 because this is consistent with the spread of the learning progression across grades. Table 7 shows the composite set of standards for perimeter and area.

- The concept of perimeter is introduced in grade 3 for regular and irregular two-dimensional shapes. The initial focus of perimeter calculations is on squares and rectangles in tandem with area. Grade 4 adds complexity to the understanding of perimeter by stressing the relationship between perimeter and area wherein students must find one dimension of a rectangle or a square given information about the area or other dimensions of the perimeter. The concept of circumference (the perimeter of a circle) is addressed in grade 6 in combination with the concept of pi and the calculation of the area of a circle.
- The development of the concept of area begins in grade 3 with the introduction of the idea of square units of measurement (square centimeters). By grade 4, the formulas for the areas of rectangles and squares are developed and understanding is deepened by standards that require students to find areas of composite figures made up of rectangles and squares. In grade 5, students are introduced to the area of nonrectangular figures with straight sides, including parallelograms, triangles, and rhombuses. Circles, including finding the area of circles, are introduced in grade 6 after students have been exposed to decimals, which is required for multiplication involving pi.

Table 7. Composite Standards: Measurement—Perimeter and Area

<table>
<tr><th></th><th>Gr. 1</th><th>Gr. 2</th><th>Gr. 3</th><th>Gr. 4</th><th>Gr. 5</th><th>Gr.6</th></tr>
<tr><td>Perimeter</td><td></td><td></td><td></td><td>• Develop the concept of perimeter
• Measure and find the perimeter of 2-dimensional shapes</td><td>• Find one dimension of a rectangle given the other dimension and area and perimeter</td><td rowspan="2">• Understand the area and the circumference of circles
• Understand the concept of pi
• Find the area and the circumference of circles, semicircles, and quarter circles</td></tr>
<tr><td>Area</td><td></td><td></td><td>• Develop the concept of area
• Compare areas, using improvised units
• Measure area in square centimeters (cm^2) and square meters (m^2)</td><td>• Apply the formula for area of squares and rectangles and composite figures made up of rectangles and squares</td><td>• Apply the formula for the area of triangles, parallelograms, and rhombuses</td></tr>
</table>

Two-dimensional shapes. The foundation for developing geometric competencies with two-dimensional shapes begins with a recognition of basic two-dimensional figures. The learning progression for two-dimensional figures then expands with the introduction of properties of angles, parallel and perpendicular lines, and symmetry. Table 8 shows the composite set of standards for two-dimensional shapes.

- The grade 1 standards focus on identifying and naming the four basic two-dimensional shapes: rectangle, square, circle, and triangle. Identification includes finding two-dimensional objects in three-dimensional shapes and completing patterns that vary according to the attributes of shape, size, and color.
- The grade 2 standards extend two-dimensional concepts to circles and semicircles. They also include having students physically copy figures on a dot grid and extend geometric patterns by identifying the orientation of shapes in addition to size and color.
- Building on students' exposure to angles and lines in grade 3, the grade 4 standards focus on properties of right angles, including rectangles and squares. Also introduced in grade 4 is the idea of symmetry of two-dimensional figures, including horizontal and vertical symmetry.
- The progression in grade 5 expands to an understanding of the application of angles in different-shaped triangles and incorporating knowledge of right angles and the sum of angles of a triangle. Also addressed is the application of angles and parallel lines to properties of parallelograms, rhombuses, and trapezoids, including sums of angles and construction.

Table 8. Composite Standards: Geometry—Two-Dimensional Shapes

Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
Four basic shapes: rectangle, square ,circle, triangle • Identify and name the four basic shapes from 2-dimensional and 3-dimensional objects, describing and classifying shapes Patterns: • Make or complete patterns with 2-dimensional cut-outs according to one or two of the following attributes: – *shape* – *size* – *color*	• Identify the basic shapes that make up a given figure • Form different 2-dimensional figures with cut-outs of – *rectangle* – *square* – *triangle* – *semicircle* – *quarter circle* • Copy figures on dot grid or square grid Patterns: • Make or complete patterns with 2-dimensional cut-outs according to one or two of the following attributes: – *shape* – *size* – *orientation* – *color*		Rectangle and square: • Understand the properties of a rectangle and a square —Find unknown angles Symmetry: • Identify symmetric figures • Determine whether a straight line is a line of symmetry of a symmetric figure and complete a symmetric figure with respect to a given horizontal or vertical line of symmetry	Triangle: • Identify and name the following types of triangles: – *isosceles triangle* – *equilateral triangle* – *right-angled triangle* • Use the property that the angle sum of a triangle is 180^{o} tp find unknown angles • Draw a triangle from given dimensions using ruler, protractor, and set squares Parallelogram, rhombus, and trapezoid: • Identify and name parallelogram, rhombus, and trapezoid • Understand the properties of parallelogram, rhombus, and trapezoid • Find unknown angles • Draw a square, rectangle,	

				parallelogram, rhombus, or trapezoid from given dimensions using ruler, protractor, and set squares	

Findings

The composite standards have a number of features that can inform an international benchmarking process for the development of K–6 mathematics standards in the U.S.

First, the composite standards *concentrate the early learning of mathematics on the numbers, measurement, and geometry strands* with less emphasis on data analysis and little exposure to algebra. The Hong Kong, China standards for grades 1–3 devote approximately half the targeted time to numbers and almost all the time remaining to geometry and measurement.

Second, the composite standards *sequence topics* within strands to support in-depth and efficient development of mathematics content following a logical development of mathematical knowledge. For example, the numbers strand sequence progression is whole numbers, arithmetic operations, fractions, decimals, ratios, and percents. Measurement introduces linear measurement followed by perimeter and area (two-dimensional measurement) and then more complicated volume (three-dimensional measurement). Geometry initially introduces the features of shapes, proceeds to cover two-dimensional geometry along with angles and parallel lines, and concludes with the features of three-dimensional figures. Data analysis starts with pictograms, a visual and more familiar way to examine data, and then moves on to bar charts and more-complicated continuous line charts.

Third, the composite standards *sequence mathematical competencies within a topic* across the grades according to a mathematically logical progression. Several illustrations occur within the numbers strand. Whole numbers are ordered by size, with grade 1 addressing numbers up to 100, grade 2 up to 1,000, grade 3 up to 10,000, and grade 4 up to 100,000. Grade 5 emphasizes an understanding of large numbers in general. Multiplication is also carefully developed, with grade 2 starting with the basic multiplication concept and multiplication tables for 2, 3, 4, 5, and 10; grade 3 extends to tables 6, 7, 8, and 9 along with multiplication of one digit by two and three digits; grade 4 introduces associative and commutative properties and multiplication of two-digit numbers by three-digit numbers; and grade 5 covers common multiples and the relation with common divisors.

Fourth, the *ordering of content for one topic is frequently aligned to reinforce the content of another topic* for the same or prior grades. Linear measurement in grade 1 introduces the centimeter, which is aligned with grade 1 whole numbers exposure of numbers up to 100. Grade 3 introduces kilometers and millimeters after 1,000 is taught within the whole numbers strand of grade 2. Grade 3 introduces the multiplication and division of money (e.g., relations between total costs with price and quantity), thus reinforcing the learning of multiplication and division in grades 2 and 3. Still another example of cross-topic reinforcement occurs within grade 6 data analysis, which introduces pie charts around the same time that circles are introduced in geometry.

In addition, it is important to note that, in many cases, particularly within the number strand, the composite standards show a grade placement of a particular skill or concept that is one year earlier than is common in much of the United States. While this is a notable finding, we believe that it is the coherent learning progressions and content connections that are much more important to emulate than the grade placement of particular topics. Furthermore, the delineation of content by learning progressions facilitates an adjustment of the grade placement of content to fit the learning pace of individual students within a common standards framework that all students are eventually expected to master.

In conclusion, standards may only be the front end of a long-term the reform process, but it is critical that sound standards be developed to guide the rest of the reform process. The composite mathematics standards of the three Asian high performers offer a theoretically and empirically valid international benchmark for the development of common U.S. standards in mathematics.

NOTE: The complete version of this paper is available at:
http://www.air.org/news/documents/MathStandards.pdf

References

Asian Pacific Economic Cooperation (APEC) (n.d.). Education Network. Available at http://hrd.apec.org/index.php/Education_Network_%28EDNET%29

Mullis, I., Martin, M., & Foy, P. (2008). *TIMSS 2007 international mathematics report: Findings from IEA's Trends in International Mathematics and Science Study at the Fourth and Eighth Grades.* Chestnut Hill, MA: Boston College, TIMSS & PIRLS International Study Center. Available online at http://timss.bc.edu/TIMSS2007/PDF/TIMSS2007_InternationalMathematicsReport.pdf.

National Assessment Governing Board. (2008). *NAEP 2009 science framework.* (ED–04–CO–0148). Washington, DC: Author. Available September 2009 at http://www.nagb.org/publications/frameworks/science-09.pdf.

Race to the Top Fund, 74 Fed. Reg. 37804–37836 (July 29, 2009)

Schmidt, W., Houang, R., & Cogan, L. (2002). A coherent curriculum. *American Educator, 26*(2), 1–17.

Appendix

Table A1. Composite Standards: Numbers and Operations

Topics	Gr. 1	Gr.2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
Whole Numbers / Place Value	Whole numbers to 100: • Count to tell the number of objects in a given set • Count forward and backward • Compare the number of objects in two or more sets • Use ordinal numbers (first, second, up to tenth) and symbols (1st, 2nd, 3rd, etc.) • Use number notation and place values (tens, ones) • Read and write numbers in numerals and in words • Compare and order numbers	Whole numbers to 1,000: • Count in tens and hundreds • Use number notation and place values (hundreds, tens, ones) • Read and write numbers in numerals and in words • Compare and order numbers	Whole numbers to 10,000: • Use number notation and place values (thousands, hundreds, tens, ones) • Read and write numbers in numerals and in words • Compare and order numbers • Understand odd and even numbers	Whole numbers to 100,000: • Use number notation and place values (ten thousands, thousands, hundreds, tens, ones) • Read and write numbers in numerals and in words • Compare and order numbers • Round numbers to the nearest 10 or 100	• Develop an understanding of large numbers • Develop the concept of approximation • Estimate the number of a large quantity of objects • Round large numbers in thousands, ten thousands, hundred thousands, millions, ten millions, hundred millions	
Addition / Subtraction	Addition and subtraction: • Understand situations for , and the meaning of, addition and subtraction	Addition and subtraction of numbers up to three digits: • Solve up to 2-step word problems involving addition	Addition and subtraction of numbers up to four digits: • Use the terms “sum” and “difference”			

Topics	Gr. 1	Gr.2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
	• Use the addition symbol (+) or the subtraction symbol (−) • Compare two numbers within 20 to tell how much one number is greater (or smaller) than the other • Recognize the relationship between addition and subtraction • Build the addition bonds up to 9 + 9 • Solve 1-step word problems involving addition and subtraction within 20 • Add more than two 1-digit numbers • Add and subtract within 100 without regrouping involving – *a 2-digit number and ones* – *a 2-digit number and tens* – *two 2-digit numbers* • Use mental	and subtraction • Use mental calculation for addition and subtraction involving – *a 3-digit number and ones* – *a 3-digit number and tens* – *a 3-digit number and hundreds*	• Solve up to 2-step word problems involving addition and subtraction			

Topics	Gr. 1	Gr.2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
	calculation for addition and subtraction – *within 20* – *involving a 2-digit number and ones without renaming* Involving a 2-digit number and tens					
Multiplication/Division		Basic multiplication (basic concept and computation): • Understand the situations for, and meaning of, multiplication • Build up the multiplication tables of 2, 3, 4, 5, and 10 • Discover the commutative property of multiplication through concrete examples (e.g., $2 \times 3 = 3 \times 2$) Basic division (basic concept and computation): • Develop the concept of division: sharing and grouping	Multiplication: • Build up the multiplication tables of 6, 7, 8, and 9 • Perform multiplication with a multiplier of 1 digit and a multiplicand of 2 or 3 digits Division: • Understand the situations for, and meaning of, division • Perform basic division by short division • Perform division with a divisor of 1 digit and a dividend of 2 or 3 digits with and without remainders	Multiplication: • Discover the associative property of multiplication through concrete examples • Apply the commutative and associative properties of multiplication in computation (e.g., $2 \times 8 \times 5 = (2 \times 5) \times 8$) • Perform multiplication with a multiplier of 2 digits and a multiplicand of 2 digits and then 3 digits Division: • Perform division with a divisor of 2 digits and a	Divisors and multiples: • Understand the meaning of “divisor,” “common divisor,” and “greatest common divisor” and know how to solve for them • Understand the meaning of “multiple,” “common multiple,” and “least common multiple” and know how to solve for them • Understand the relation between divisors and multiples and know how to apply them • Multiply and divide by 10, 100 and	

Topics	Gr. 1	Gr.2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
		• Divide a quantity (not greater than 20) into equal sets given – *the number of objects in each set* – *the number of sets* • Recognize the relationship between multiplication and division • Solve 1-step word problems involving multiplication and division within the multiplication tables	• Use the terms "product," "quotient," and "remainder" • Solve up to 2-step word problems involving the four operations, including estimating answers.	dividend of 2 and then 3 digits • Recognize divisibility when the divisors are 2, 5, and 10 • Identify 1-digit factors of 2-digit numbers • Distinguish between factors and multiples • Solve up to 3-step word problems involving the four operations, including estimating answers	1000 mentally • Use order of operations, combined operations involving the four operations, and brackets • Solve word problems involving the four operations, including estimating answers	
Fractions / Concepts		Fraction of a whole: • Interpret a fraction as part of a whole • Read and write fractions • Compare and order unit fractions and like fractions. (denominators less than or equal to 12)	Equivalent fractions: • Recognize and name equivalent fractions • Write the equivalent fraction of a fraction, given the denominator or the numerator • Express a fraction in its simplest form • Compare and order unlike fractions, including comparing fractions with respect to one half	Mixed numbers and improper fractions: • Understand the concepts of mixed numbers and improper fractions • Express an improper fraction as a mixed number, and vice versa, and expressing both in simplest form		

Topics	Gr. 1	Gr.2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
			(denominators less than or equal to 12)			
Fractions/ Arithmetic Operations			Addition and subtraction of two related fractions (one denominator a factor of the other) within one whole (denominators of given fractions should not exceed 12)	Addition and subtraction of • like fractions • related fractions (denominators of given fractions should not exceed 12) Multiplication of a proper or improper fraction and a whole number	Addition and subtraction of fractions with unlike denominators: • Add and subtract fractions with unlike denominators Multiplication of fractions: • proper fractions, improper fractions, mixed numbers and whole numbers by proper fractions, improper fractions and mixed numbers • Divide fractions by whole numbers and whole numbers by fractions	Division of fractions: • Divide proper fractions by proper fractions Mixed calculations with fraction and decimal: • Know how to solve simple calculations with both fractions and decimals
Decimals				Decimals up to three decimal places: • Understand notation and place values (tenths, hundredths, thousandths), including identifying the values of the digits in a decimal	Decimal addition and subtraction: • Add and subtract decimals up to two places of decimals and for sums involving at most three operations —Estimate the answers	Decimal division: • Develop an understanding of division of decimals through daily life examples • Divide decimals by whole numbers, whole numbers by decimals, and

Topics	Gr. 1	Gr.2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
				• Use the number line to display decimals • Compare and order decimals • Convert a decimal to a fraction • Convert a fraction whose denominator is a factor of 10 or 100 to a decimal • Round decimals to the nearest whole number	Decimal multiplication: • Develop an understanding of multiplication of decimals through daily life examples • Multiply decimals by whole numbers and by decimals • Estimate the answers	decimals by decimals • Perform mixed operations on decimals for sums involving at most three operations • Estimate the answers Decimal conversion: • Convert decimals into fractions and fractions to decimals • Compare fractions by converting them into decimals • Estimate the answers
Ratio					Ratio (excludes ratios involving fractions and decimals): • Interpret *a* : *b* and *a* : *b* : *c*, where *a*, *b*, and *c* are whole numbers • Express a ratio in its simplest form • Find the ratio of two or three given quantities • Write equivalent ratios and find the missing term in a pair of equivalent ratios	

Topics	Gr. 1	Gr.2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
					• Solve up to 2-step word problems involving ratio, including finding one quantity given the other quantity and their ratio	
Percents						Percents (basic concept; conversion of percentages into decimals or fractions and vice versa): • Recognize percentages through daily life examples • Develop an understanding of percentages • Convert percentages into decimals and vice versa Applications of percents: • Solve simple problems on percentages, including finding percentages expressing the value of a percentage of a quantity applying discounts • Estimate the

Topics	Gr. 1	Gr.2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
						answers
NCTM	Grade 1. Number and Operations: Developing an understanding of whole number relationships including grouping in tens and ones. Children compare and order whole numbers (at least to 100) to develop an understanding of and solve problems involving the relative sizes of these numbers. They think of whole numbers between 10 and 100 in terms of groups of tens and ones (especially recognizing the numbers 11 to 19 as 1 group of ten and particular numbers of ones). They understand the sequential order of the counting numbers and their relative magnitudes and represent numbers on a number line. Number and Operations and Algebra: Developing understandings of addition and subtraction and strategies for basic addition facts and related subtraction facts. Children develop strategies for adding and subtracting whole numbers on the basis of their earlier work with small numbers. They use a variety of models, including discrete objects, length-based models (e.g., lengths of connecting cubes), and number lines, to model "part-whole," "adding to," "taking away from," and "comparing" situations to develop an understanding of the meanings of addition and subtraction and strategies to solve such arithmetic problems. Children understand the connections between counting and the operations of addition and subtraction (e.g., adding two is the same as "counting on" two). They use properties of addition (commutativity and associativity) to add whole numbers, and they create and use increasingly sophisticated strategies based on these properties (e.g., "making tens") to solve addition and subtraction problems involving basic facts. By comparing a variety of solution strategies, children relate addition and subtraction as inverse operations. Grade 2. Number and Operations: Developing an understanding of the base-ten numeration system and place-value concepts. Children develop an understanding of the base-ten numeration system and place-value concepts (at least to 1000). Their understanding of base-ten numeration includes ideas of counting in units and multiples of hundreds, tens, and ones, as well as a grasp of number relationships, which they demonstrate in a variety of ways, including comparing and ordering numbers. They understand multidigit numbers in terms of place value, recognizing that place-value notation is a shorthand for the sums of multiples of powers of 10 (e.g., 853 as 8 hundreds + 5 tens + 3 ones). Number and Operations and Algebra: Developing quick recall of addition facts and related subtraction facts and fluency with multidigit addition and subtraction. Children use their understanding of addition to develop quick recall of basic addition facts and related subtraction facts. They solve arithmetic problems by applying their understanding of models of addition and subtraction (such as combining or separating sets or using number lines), relationships and properties of number (such as place value), and properties of addition (commutativity and associativity).Children develop, discuss, and use efficient, accurate, and generalizable methods to add and subtract multidigit whole numbers. They select and apply appropriate methods to estimate sums and differences or calculate them mentally, depending on the context and numbers involved. They develop fluency with efficient procedures, including standard algorithms, for adding and subtracting whole numbers, understand why the procedures work (on the basis of place value and properties of operations), and use them to solve problems. Grade 3. Number and Operations and Algebra: Developing understandings of multiplication and division and strategies for basic multiplication facts and related division facts. Students understand the meanings of multiplication and division of whole numbers through the use of representations (e.g., equal-sized groups, arrays, area models, and equal "jumps" on number lines for multiplication, and successive subtraction, partitioning, and sharing for division). They use properties of addition and multiplication (e.g., commutativity, associativity, and the distributive property) to multiply whole numbers and apply increasingly sophisticated strategies based on these properties to solve multiplication and division problems involving basic facts. By comparing a variety of solution strategies, students relate multiplication and division as inverse						

Topics	Gr. 1	Gr.2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
	operations. Number and Operations: Connecting ratio and rate to multiplication and division Students use simple reasoning about multiplication and division to solve ratio and rate problems (e.g., “If 5 items cost $3.75 and all items are the same price, then I can find the cost of 12 items by first dividing $3.75 by 5 to find out how much one item costs and then multiplying the cost of a single item by 12”). By viewing equivalent ratios and rates as deriving from, and extending, pairs of rows (or columns) in the multiplication table, and by analyzing simple drawings that indicate the relative sizes of quantities, students extend whole number multiplication and division to ratios and rates. Thus, they expand the repertoire of problems that they can solve by using multiplication and division, and they build on their understanding of fractions to understand ratios. Students solve a wide variety of problems involving ratios and rates. Grade 4. Number and Operations and Algebra: Developing quick recall of multiplication facts and related division facts and fluency with whole number multiplication. Students use understandings of multiplication to develop quick recall of the basic multiplication facts and related division facts. They apply their understanding of models for multiplication (i.e., equal-sized groups, arrays, area models, equal intervals on the number line), place value, and properties of operations (in particular, the distributive property) as they develop, discuss, and use efficient, accurate, and generalizable methods to multiply multidigit whole numbers. They select appropriate methods and apply them accurately to estimate products or calculate them mentally, depending on the context and numbers involved. They develop fluency with efficient procedures, including the standard algorithm, for multiplying whole numbers, understand why the procedures work (on the basis of place value and properties of operations), and use them to solve problems. Number and Operations: Developing an understanding of decimals, including the connections between fractions and decimals. Students understand decimal notation as an extension of the base-ten system of writing whole numbers that is useful for representing more numbers, including numbers between 0 and 1, between 1 and 2, and so on. Students relate their understanding of fractions to reading and writing decimals that are greater than or less than 1, identifying equivalent decimals, comparing and ordering decimals, and estimating decimal or fractional amounts in problem solving. They connect equivalent fractions and decimals by comparing models to symbols and locating equivalent symbols on the number line. Grade 5. Number and Operations and Algebra: Developing an understanding of and fluency with division of whole numbers. Students apply their understanding of models for division, place value, properties, and the relationship of division to multiplication as they develop, discuss, and use efficient, accurate, and generalizable procedures to find quotients involving multidigit dividends. They select appropriate methods and apply them accurately to estimate quotients or calculate them mentally, depending on the context and numbers involved. They develop fluency with efficient procedures, including the standard algorithm, for dividing whole numbers, understand why the procedures work (on the basis of place value and properties of operations), and use them to solve problems. They consider the context in which a problem is situated to select the most useful form of the quotient for the solution, and they interpret it appropriately. Number and Operations: Developing an understanding of and fluency with addition and subtraction of fractions and decimals. Students apply their understandings of fractions and fraction models to represent the addition and subtraction of fractions with unlike denominators as equivalent calculations with like denominators. They apply their understandings of decimal models, place value, and properties to add and subtract decimals. They develop fluency with standard procedures for adding and subtracting fractions and decimals. They make reasonable					

Topics	Gr. 1	Gr.2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
	estimates of fraction and decimal sums and differences. Students add and subtract fractions and decimals to solve problems, including problems involving measurement. Grade 6. Number and Operations: Developing an understanding of and fluency with multiplication and division of fractions and decimals. Students use the meanings of fractions, multiplication and division, and the inverse relationship between multiplication and division to make sense of procedures for multiplying and dividing fractions and explain why they work. They use the relationship between decimals and fractions, as well as the relationship between finite decimals and whole numbers (i.e., a finite decimal multiplied by an appropriate power of 10 is a whole number), to understand and explain the procedures for multiplying and dividing decimals. Students use common procedures to multiply and divide fractions and decimals efficiently and accurately. They multiply and divide fractions and decimals to solve problems, including multistep problems and problems involving measurement. Number and Operations: Connecting ratio and rate to multiplication and division. Students use simple reasoning about multiplication and division to solve ratio and rate problems (e.g., "If 5 items cost $3.75 and all items are the same price, then I can find the cost of 12 items by first dividing $3.75 by 5 to find out how much one item costs and then multiplying the cost of a single item by 12"). By viewing equivalent ratios and rates as deriving from, and extending, pairs of rows (or columns) in the multiplication table, and by analyzing simple drawings that indicate the relative sizes of quantities, students extend whole number multiplication and division to ratios and rates. Thus, they expand the repertoire of problems that they can solve by using multiplication and division, and they build on their understanding of fractions to understand ratios. Students solve a wide variety of problems involving ratios and rates.					

Table A2. Composite Standards: Measurement

	Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr.6
Linear measurement:						
Concept of length	• Understand concepts of length and distance • Understand long, longer, longest, short, shorter, shortest • Understand centimeter	• Understand that a meter is greater than a centimeter	• Understand that a kilometer is greater than a meter and that a millimeter is smaller than a centimeter • Understand that—Convert compound units to a smaller or a larger unit		• Convert from a smaller unit to a larger unit and vice versa in decimal form	
Tools/ measuring length	• Measure and compare lengths of objects and distance with centimeters • Estimate lengths and distances • Measure length with appropriate tools	• Measure and compare length and distance in meters and centimeters • Estimate lengths and distances • Measure length with appropriate tools	• Measure and compare length and distance in kilometers and millimeters • Estimate lengths and distances • Measure length with appropriate tools			
Perimeter / Area						
Perimeter				• Develop the concept of perimeter • Measure and find the perimeter of 2-dimensional shapes	• Find one dimension of a rectangle given the other dimension and area and perimeter	• Understand the area and the circumference of circles • Understand the concept of pi • Find the area and the circumference of circles, semicircles, and quarter circles
Area			• Develop the concept of area • Compare areas, using improvised units • Measure area in	• Apply the formula for area of squares and rectangles and composite figures made up of rectangles and	• Apply the formula for the area of triangles, parallelograms, and rhombuses	

	Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr.6
			square centimeters (cm^2) and square meters (m^2)	squares		
Volume						
Concept			• Develop the concept of capacity and volume • Understand the need for standardized units of measurement • Understand the units of liter and milliliter		• Introduce the standard unit cubic centimeter (cm^3) • Understand the need for using a unit larger than a cubic centimeter: cubic meter (m^3)	
Tools / Measuring			• Measure and compare the volumes of containers using liter and milliliter • Measure volume with appropriate tools		• Measure and compare objects using cm^3 • Understand and apply the formula for finding the volume of cubes and cubes	• Find the length of one edge of a cube given its volume or find the height of a cube given its volume and base area
Nongeometric Measurement						
Time: Clock	• Tell and write time to the hour and half hour	• Tell and write time to 5 minutes • Use a.m. and p.m.	• Tell and write time to 1 minute • Solve word problems involving adding and subtracting time down to the minute	• Measure time in seconds • Use a 24-hour clock to solve clock word problems		
Time: Calendar	• Learn the days of the week • Recognize that there are 12 months in a year	• Recognize the number of days in a month and a year • Understand the relation among 1 hour, 1 day, 1				

	Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr.6
		week, 1 month, and 1 year				
Money	• Tell the amount of money in cents up to $1 and in dollars up to $100 (Excludes combination of dollars and cents)	• Read and write money in decimal notation	• Solve problems involving adding and subtracting money in decimal notation	• Solve word problems involving the four operations and money in decimal notation		
Weight	• Develop the concept of weight • Compare the weights of concrete objects • Express heavy and light and use the terms “heavy,” “heavier,” and “heaviest”	• Understand the need for using standard units • Measure and compare the weights of objects using gram (g) and kilogram (kg) • Choose the appropriate tools for measuring			• Convert a measurement from a smaller unit to a larger unit in decimal form and vice versa, using kilograms and grams	
NCTM	*Measurement* and *Data Analysis: (Grade 1)* Children strengthen their sense of number by solving problems involving measurements and data. Measuring by laying multiple copies of a unit end to end and then counting the units by using groups of tens and ones supports children’s understanding of number lines and number relationships. Representing measurements and discrete data in picture and bar graphs involves counting and comparisons that provide another meaningful connection to number relationships. *Measurement:* Developing an understanding of area and determining the areas of two dimensional Shapes (Grade 3) Students recognize area as an attribute of two-dimensional regions. They learn that they can quantify area by finding the total number of same-sized units of area that cover the shape without gaps or overlaps. They understand that a square that is 1 unit on a side is the standard unit for measuring area. They select appropriate units, strategies (e.g., decomposing shapes), and tools for solving problems that involve estimating or measuring area. Students connect area measure to the area model that they have used to represent multiplication, and they use this connection to justify the formula for the area of a rectangle. *Geometry* and *Measurement* and *Algebra:* Describing three-dimensional shapes and analyzing their properties, including volume and surface area (Grade 5) Students relate two-dimensional shapes to three-dimensional shapes and analyze properties of polyhedral solids, describing them by the number of edges, faces, or vertices as well as the types of faces. Students recognize volume as an attribute of three-dimensional space. They understand that they					

	Gr. 1	Gr. 2	Gr. 3	Gr. 4	Gr. 5	Gr.6
	can quantify volume by finding the total number of same-sized units of volume that they need to fill the space without gaps or overlaps. They understand that a cube that is 1 unit on an edge is the standard unit for measuring volume. They select appropriate units, strategies, and tools for solving problems that involve estimating or measuring volume. They decompose three-dimensional shapes and find surface areas and volumes of prisms. As they work with surface area, they find and justify relationships among the formulas for the areas of different polygons. They measure necessary attributes of shapes to use area formulas to solve problems.					

Table A3. Composite Standards: Geometry

Topics	Gr. 1	Gr.2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
2D- Shapes	Four basic shapes: rectangle, square ,circle, triangle • Identify and name the four basic shapes from 2-dimensional and 3-dimensional objects, describing and classifying shapes Patterns: • Make or complete patterns with 2-dimensional cut-outs according to one or two of the following attributes: – *shape* – *size* – *color*	• Identify the basic shapes that make up a given figure • Form different 2-dimensional figures with cut-outs of – *rectangle* – *square* – *triangle* – *semicircle* – *quarter circle* —Copy figures on dot grid or square grid Patterns: • Make or complete patterns with 2-dimensional cut-outs according to one or two of the following attributes: – *shape* – *size* – *orientation* – *color*		Rectangle and square: • Understand the properties of a rectangle and a square —Find unknown angles Symmetry: • Identify symmetric figures • Determine whether a straight line is a line of symmetry of a symmetric figure and complete a symmetric figure with respect to a given horizontal or vertical line of symmetry	Triangle: • Identify and name the following types of triangles: – *isosceles triangle* – *equilateral triangle* – *right-angled triangle* • Use the property that the angle sum of a triangle is 180^o tp find unknown angles • Draw a triangle from given dimensions using ruler, protractor, and set squares Parallelogram, rhombus, and trapezoid: • Identify and name parallelogram, rhombus, and trapezoid • Understand the properties of parallelogram, rhombus, and trapezoid • Find unknown angles • Draw a square, rectangle,	

Topics	Gr. 1	Gr.2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
					parallelogram, rhombus, or trapezoid from given dimensions using ruler, protractor, and set squares	
3-D Shapes	• Recognize prisms, pyramids, and spheres. • Identity 3-dimensional shapes intuitively • Group 3-dimensional shapes • Describe the relative positions of two 3-dimensional shapes briefly • Make or complete patterns with 3-dimensional models, including, cube (rectangular block), cone, and cylinder	• Identify prisms, cylinders, pyramids and cones • Recognize faces • —Group 3-dimensional shapes • Make 3-dimensional shapes • Form different 3-dimensional figures with concrete models of – *cube* – *cone* – *cylinder*				• Understand the concepts of prisms and pyramids and their components and properties • Work with various solid figures: – *Looking at a solid figure made by building blocks, count the number of blocks used* – *Make various shapes using building blocks, and find the patterns* – *Express the shape of a solid figure made by building blocks from the top, front, and side* • Identify nets of the following solids: cube, prism, and pyramid and make

Topics	Gr. 1	Gr.2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
						3-dimensional solids from given nets • Understand the concepts of cylinders and cones and their components and properties • Understand the concept of a solid of revolution
Lines		• Identify lines (straight lines) and curves • Identify edges and faces of a 3-dimensional object	• Identify and draw perpendicular and parallel lines			
Angles			• Identify angle as an amount of turning • Identify angles in 2-dimensional and 3-dimensional objects • Identify right angles, angles greater than/smaller than a right angle	• Use notation such as ∠ABC and ∠x to name angles • Estimate and measure angles in degrees • Draw an angle using a protractor • Associate 1/4 turn/right angle with 90°; 1/2 turn with 180°; 3/4 turn with 270°; complete turn with 360°; and 8-point compass		
NCTM	Grade 1. Geometry: Composing and decomposing geometric shapes Children compose and decompose plane and solid figures (e.g., by putting two congruent isosceles triangles together to make a rhombus), thus building an understanding of part-whole relationships as well as the properties of					

Topics	Gr. 1	Gr.2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
	the original and composite shapes. As they combine figures, they recognize them from different perspectives and orientations, describe their geometric attributes and properties, and determine how they are alike and different, in the process developing a background for measurement and initial understandings of such properties as congruence and symmetry. Grade 3. Geometry: Describing and analyzing properties of two-dimensional shapes Students describe, analyze, compare, and classify two-dimensional shapes by their sides and angles and connect these attributes to definitions of shapes. Students investigate, describe, and reason about decomposing, combining, and transforming polygons to make other polygons. Through building, drawing, and analyzing two-dimensional shapes, students understand attributes and properties of two-dimensional space and the use of those attributes and properties in solving problems, including applications involving congruence and symmetry. Grade 5. Geometry and Measurement and Algebra: Describing three-dimensional shapes and analyzing their properties, including volume and surface area Students relate two-dimensional shapes to three-dimensional shapes and analyze properties of polyhedral solids, describing them by the number of edges, faces, or vertices as well as the types of faces. Students recognize volume as an attribute of three-dimensional space. They understand that they can quantify volume by finding the total number of same-sized units of volume that they need to fill the space without gaps or overlaps. They understand that a cube that is 1 unit on an edge is the standard unit for measuring volume. They select appropriate units, strategies, and tools for solving problems that involve estimating or measuring volume. They decompose three-dimensional shapes and find surface areas and volumes of prisms. As they work with surface area, they find and justify relationships among the formulas for the areas of different polygons. They measure necessary attributes of shapes to use area formulas to solve problems.					

Table A4. Composite Standards: Data Analysis

Topics	Gr. 1	Gr.2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
Classifying Objects	• Classify objects or people by a predetermined standard, and count the numbers in each category.					
Pictograms		• Compare the quantity of three or more types of objects by arranging them in lines • Read, construct and interpret picture graphs with scales • Solve problems using information presented in picture graphs				
Bar Graphs			• Read/discuss block graphs in which 1 square represents 1 unit, average value • Read, construct and interpret bar graphs in both horizontal and vertical forms, including using their scales • Solve problems using information presented in bar graphs			

Topics	Gr. 1	Gr.2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
Tables				• Complete a table from given data • Read and interpret tables • Solve problems using information presented in tables		
Line Graphs				• Collect data of continuous variates and express them in a graph of broken lines • Compare bar graphs and the graphs of broken lines to understand the properties and uses of each graph		
Averages					• Interpret average as “total amount ÷ number of items” • Calculate the average number/quantity • Solve word problems involving average, including finding the total amount given the average and the number of items	

Topics	Gr. 1	Gr.2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
Pie Charts						• Read and interpret pie charts • Solve 1-step problems using information presented in pie charts
NCTM Focal Points	Gr. 1. Measurement and Data Analysis: Children strengthen their sense of number by solving problems involving data. Representing measurements and discrete data in picture and bar graphs involves counting and comparisons that provide another meaningful connection to number relationships. Gr. 3. Addition, subtraction, multiplication, and division of whole numbers come into play as students construct and analyze frequency tables, bar graphs, picture graphs, and line plots and use them to solve problems. Gr. 4. Students continue to use tools from grade 3, solving problems by making frequency tables, bar graphs, picture graphs, and line plots. They apply their understanding of place value to develop and use stem-and-leaf plots. Gr. 5. Students apply their understanding of whole numbers, fractions, and decimals as they construct and analyze double-bar and line graphs and use ordered pairs on coordinate grids.					

Table A5. Composite Standards: Algebra

Topics	Gr. 1	Gr.2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
Expressions					Expressions: • Use symbols or letters to represent numbers • Record with algebraic symbols, for example, "John is *x* years old now. How old will he be after 10 years?" and record as (*x* + 10) years old	
Equations					Simple equations involving 1 step in finding the solution: • Understand the concept of equations • Solve simple equations involving 1 step in the solutions and check the answers (involving whole numbers only) • Solve problems by simple equations (involving only 1 step in the solutions)	Simple equations (involving 2 steps in finding the solution): • Solve equations involving at most 2 steps in the solutions, and examine the results • Solve problems by using simple equations (involving at most 2 steps in the solution)
NCTM Focal Points	Grade 6. Writing, interpreting, and using mathematical expressions and equations Students write mathematical expressions and equations that correspond to given situations, they evaluate expressions, and they use expressions and formulas to solve problems. They understand that variables represent numbers whose exact values are not yet specified, and they use variables appropriately. Students understand that expressions in different forms can be equivalent, and they can rewrite an expression to represent a quantity in a different way (e.g., to make it more compact or to feature different information). Students know that the solutions of an equation are the values of the variables that make the equation true. They solve simple					

Topics	Gr. 1	Gr.2	Gr. 3	Gr. 4	Gr. 5	Gr. 6
	one-step equations by using number sense, properties of operations, and the idea of maintaining equality on both sides of an equation. They construct and analyze tables (e.g., to show quantities that are in equivalent ratios), and they use equations to describe simple relationships (such as 3x = y) shown in a table.					

Grades 7-12 Learning Progressions in Mathematics Content

Zalman Usiskin, The University of Chicago

March 2010

Originally written for the APEC Conference
on Exemplary Practices in Mathematics Education
Koh Samui, Thailand, 8-12 March 2010
Expanded after the conference based on suggestions there.

In grades K-6, we teach primarily arithmetic and simple geometry for one basic reason: as part of basic literacy. At the secondary level, we teach mathematics, or, perhaps more accurately, the *mathematical sciences* (including statistics, computer science, operations research, et al.) still as part of basic literacy but for several other major reasons: to be a wise consumer; to be an informed citizen capable of understanding issues of the day; to apply on the job; and (for only a few) to make discoveries that will expand the field. Because of its importance, throughout the world mathematics enjoys a status in schools second only to reading and writing in one's native language.

The mathematics curriculum has many sizes. From smallest to largest, they are:

the problem or episode	a few seconds to many minutes
the lesson	a class period or two
the chapter or unit	a few weeks
the course	typically, a half year or year
the school mathematics curriculum	K-12
the entire school curriculum	K-12 (all subjects)

The phrase "learning progressions in mathematics content" suggests "big ideas" that are at the size of the course or the school mathematics curriculum. That is, for the most part, these ideas take many months or several years to develop. These are the ideas that I discuss in this paper. But the good curriculum and the good teacher make many smaller progressions, often within an individual lesson, sometimes within a chapter or unit, and sometimes over an entire year.

Although this paper is concerned mainly with the specific grade range 7-12 (i.e., what is often termed *secondary education*), some of the learning progressions described here should typically begin in primary education earlier than grade 7, while others will go past grade 12 into tertiary education. Many of the progressions described here have been applied in developing the materials for grades 6-12 of the University of Chicago School Mathematics Project, but a casual look at the materials will usually not uncover the progressions because they tend to be beneath the surface.

My list contains nine progressions, not ordered by any measure of importance. It could have contained a few more. The list purposely ignores the sequencing of algorithms and the questions of the order of deduction in geometry. Algorithmic and logical sequences have formed the basis for virtually all school mathematics instruction over the years and are so familiar to mathematics educators that their repetition is not needed. However, I believe they should be seen in a particular perspective relative to the nine learning progressions, so I comment on them towards the end of this paper. Also, some progressions of slightly lesser importance have been omitted for lack of time and space.

Progression 1: from whole number to rational number to real number, and then to complex number and vector

I begin here because this progression is the earliest in schooling, beginning even before school. The key question here is: *What are the numerical objects of mathematics*?

The numerical objects obviously begin with the counting numbers, including 0. Early in the primary grades, through measurement and money, students should see that the counting numbers do not suffice and that negative numbers are natural in situations that have two opposite directions, such as above and below sea level, or profit and loss. By grade 7, we hope that the student realizes that symbols such as 116.42 and $\frac{5}{12}$ represent single numbers, not two numbers 5 and 12 with a mysterious bar between them, and that students also see 5 and +5 as the same, and -5 as a single number and not as a number with a mysterious sign.

The move from fractions and decimals to the rational numbers requires that students understand that the rational numbers are dense, that is, that we can find rational numbers as close to a given rational number as we like, both greater than it and less than it. The number line is a powerful representation for this move as it also is for the indication that a single number may be represented in a variety of ways.

These ideas are necessary for the progression to irrational numbers and real numbers, and for the idea of continuity that students will encounter in their study of functions. There are several ways to get from rationals to irrationals. One common way is via nested intervals. For $\sqrt{2}$, we square rational numbers to see if their squares are less than or greater than 2.

$1^2 < 2$ and $2 < 2^2$, so $1 < \sqrt{2} < 2$.

$1.4^2 < 2$ and $2 < 1.5^2$, so $1.4 < \sqrt{2} < 1.5$.

$1.41^2 < 2$ and $2 < 1.42^2$, so $1.41 < \sqrt{2} < 1.42$.

…and so on. And we conclude that $\sqrt{2}$ is described by a decimal that begins 1.41…

A problem with this sequence is that students come to think that real numbers *are* decimals rather than that they *can be represented by* decimals. So it is important to get at some real numbers directly. We can do it easily with $\sqrt{2}$ by noting that this is the length of a diagonal of a unit square. I have colleagues who have trouble with this idea; they think that lengths are really rational numbers because in everyday life we compute with rational numbers and not irrationals. I argue that since the the length of the diagonal of a real square is as close to $\sqrt{2}$ as the length of its side is to

1. A similar argument can be made for π and many other irrationals. We can then move to the notion that any infinite decimal represents a number.

This progression can branch from real numbers in three ways. A first branch is to vector. If numerical objects can represent points on a number line, then why not points in the plane? Addition and subtraction of vectors and the cross product for multiplication help students to appreciate the properties of operations of numbers with which they are familiar. And if objects can represent points on a line or points in a plane, then why not points in space? In school mathematics, we do not go beyond 3-dimensional vectors, but this progression continues into the study of linear algebra in college.

A second branch is to complex number. Although it is common and natural to introduce complex numbers as solutions to polynomial equations that cannot be solved in the reals, a disadvantage of this order is that students too easily interpret the terms *real* and *imaginary* as descriptors chosen because the set of numbers they describe *exist* and *do not exist*.

The concreteness of the complex numbers can come to play earlier if these numbers are associated with the coordinate plane as reals are with the number line, and their addition and multiplication are seen geometrically to generalize addition and multiplication of reals. One of the most interesting aspects of this connection between arithmetic and geometry is the fact that addition of complex numbers is easy in rectangular coordinates (that is, $(a, b) + (c, d) = (a + c, b + d)$), while multiplication is easy in polar coordinates ($[r, \theta] \bullet [s, \phi] = [rs, \theta + \phi]$), and turns DeMoivre's Theorem into a corollary.

Throughout this particular path of this progression from whole number to complex number, a student shouold view the arithmetic operations as being able to be interpreted both as binary operations (e.g., adding two numbers yields a third number) and as unary operations (adding by a particular constant number has its own properties). E.g., it is as unary operations that students learn what it means to add 0 to a number, and that adding *n* and subtracting *n* are inverse operations.

A third branch from real number is to the matrix as an object that can represent a single point, a finite set of points, a vector, or more generally, multi-dimensional data. Many students do not understand the properties of the operations of arithmetic until they have seen objects such as matrices for which important operations of addition and multiplication can be defined but do not possess all the field properties. The connection of matrices with vectors, which can wait until the tertiary level, brings the first and third branches together.

Progression 2: from numerical expression to algebraic expression, and then to function as a relationship and then to function as an object

The progression from numerical expression to algebraic expression includes with it the progression from *number* to *variable*.

In the primary school, the student should be introduced to two uses of the idea of *variable*: (1) variable as *unknown*, as in 3 + ___ = 10 or 3 + x = 10; (2) variable as *generalized arithmetic*, as in

A = LW as describing (area of a rectangle = length times width), or

$a + 0 = a$ as generalizing the instances (9.6 + 0 = 9.6) and ($\frac{2}{3} + 0 = \frac{2}{3}$).

In the secondary school, the student then can be introduced to a third important use: (3) variable as *function argument* or *parameter*, as in f: x $\rightarrow$ 3x + 5, in which the idea of a variable continuously varying first appears. At the tertiary level, a fourth use, (4) variable as *arbitrary symbol*, as in descriptions of the 4-group {I, *a*, *b*, *ab*} by the equations $a^2 = b^2 = \mathrm{I}$; $ab = ba$ together with group properties.

A key idea in the progression from counting number to rational number is the treatment of a fraction as a single number. This idea, that a pair or larger group of symbols can be viewed as one, is called *chunking* by psychologists; it is the cognitive mechanism by which we view a string of letters as a single word, the cognitive mechanism underlying all of reading, and it is exceedingly important in the progression from arithmetic expression to algebraic expression.

Let us consider a common pattern used in textbooks, such as the length of a train in which the engine at the front is 30 meters and each car is 20 meters long. We ask for the length of the train. At the primary school level, students make a table.

Number of cars	Length of train
1	30 + 20 = 50
2	30 + 20 + 20 = 70
3	30 + 20 + 20 + 20 = 90
…	…

The next step in the progression is to convert the repeated additions to multiplication.

Number of cars	Length of train	Length of train
1	30 + 20 = 50	30 + 20 = 50
2	30 + 20 + 20 = 70	30 + 2•20 = 70
3	30 + 20 + 20 + 20 = 90	30 + 3•20 = 90
…	…	…

It is critical that the student understand the importance of the expressions 30 + 20, 30 + 2•20, and 30 + 3•20. They are not just for calculating the answer. They are for generating a pattern that will enable a person to find quickly the length of the train regardless of how many cars there are. A slight change in the table helps.

Number of cars	Length of train	Length of train
1	30 + 20 = 50	30 + **1**•20 = 50
2	30 + 20 + 20 = 70	30 + **2**•20 = 70
3	30 + 20 + 20 + 20 = 90	30 + **3**•20 = 90
…	…	…
n		30 + **n**•20

We now have an expression for the length of the train, using the variable to generalize the arithmetic. The expression represents a single number, the length, but also tells us how that length was calculated. We graph the ordered pairs (n, 20n + 30) as dots and find that the dots lie on a line. We have pictured a function. At this time, this function is a relationship – given the input *n*, the expression indicates the output 30 + 20*n*. When we write f(n) = 30 + 20n, we reinforce that idea. We are now using the variable *n* as an argument in a function. Only when the student sees and graphs many other relationships, noting that some are linear and some are not, does it make sense to try to categorize functions into linear, quadratic, etc. And then, when we look at the properties of these functions, it makes sense that we can name a function by a single letter. If we have used f(n) notation, the letter to use is naturally f.

This progression, from numerical relationship between pairs of numbers to thinking of the relationship as a single object, can take years, stretching from early to late secondary school and often into the tertiary level of mathematics study. It is helped by having operations on functions, such as function addition or function composition. The move to thinking of functions as objects requires that we have properties of classes of functions that are not the same as properties of individual functions. For instance, to assert that the set of linear functions is closed under composition requires that a student think of a function as an object. In my opinion, the practice of some mathematicians and in some technology that f(x) stands for a function hurts this progression. To me it is important to distinguish between the value of a function and the function itself.

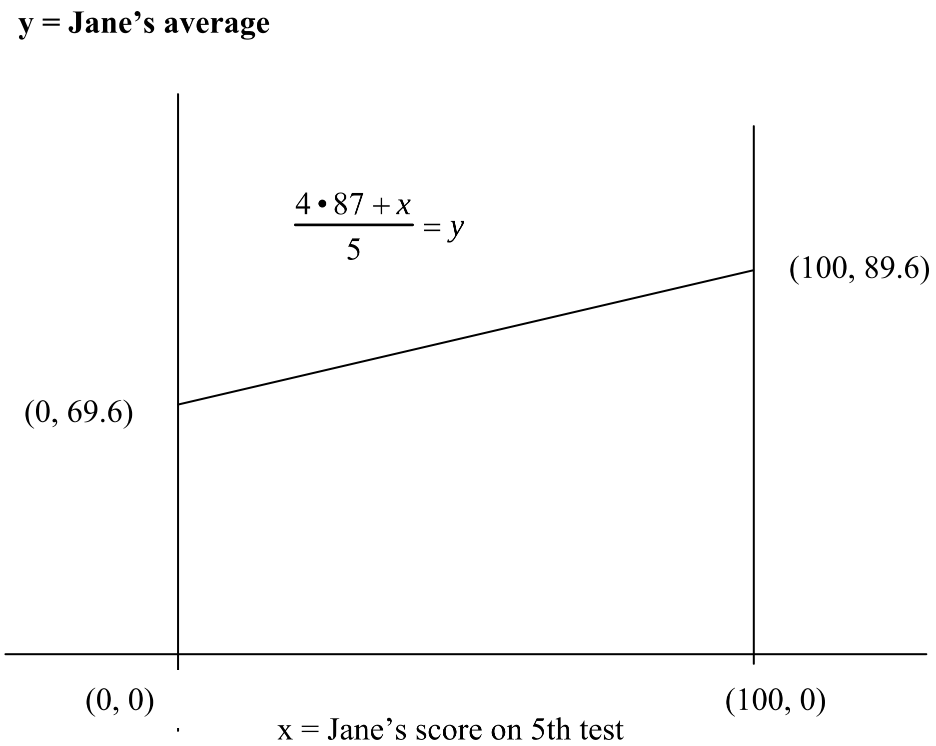
Here is another illustrative example that begins with a typical problem and shows how the progression is often poorly made.

Jane has an average of 87 after 4 tests. What score does she need on the 5th test to average 90 for all five tests?

When this question is given along with the study of algebra, the student is expected to let a variable such as x stand for Jane's score on the 5th test and to solve an equation such as $\frac{4 \bullet 87 + x}{5} = 90$. Here the variable is an unknown. But most students (and I have found, most teachers – even those with substantial mathematical knowledge) use arithmetic to solve the problem. This exposes a fundamental difficulty. Since the problem can be so easily solved without algebra, students naturally wonder why algebra is needed to find the unknown. Thus, though one reason for presenting this problem in a class is to show the power of algebra, the effect is the opposite. *Certain common problems that are supposed to help the progression from arithmetic to algebra actually hinder it.*

If we stop with just the solution to this problem, then we have shown that algebra is not needed, but most teachers do stop once they have the answer. To justify the use of algebra, we can generalize the problem. In the statement of the problem, replace 90 by y. (This is an easy step for us but certainly not for all students unless they have had some instruction.) If Jane's average for all 5 tests is y, then $\frac{4 \bullet 87 + x}{5} = y$. This is an equation for a linear function with slope $\frac{1}{5}$ and y-intercept $\frac{4 \bullet 87}{5}$ or 69.6. It shows that any point

Jane gets on any test contributes $\frac{1}{5}$ to her average. The graph of this equation for $0 \le x \le 100$, is a segment from (0, 69.6) to (100, 89.6), shows all the possible solutions.



By generalizing the pattern, we have now seen the power of algebra to solve an entire set of problems at once, something that arithmetic cannot do. And we have also changed the idea of x being an unknown to x and y being pattern generalizers and finally to x being an argument of a function.

Progression 3: from properties of individual figures to general properties of all figures in a particular class

I use the phrase "class of figures" because the most obvious examples are geometric, but could just have easily used "set of objects". Breaking a set of objects into various subsets based on properties is a very important idea in mathematics; we classify numbers, functions, 2-dimensional geometric figures, 3-dimensional figures, matrices, transformations, etc. Our reason for doing so is because we want to deal with properties that are held by all objects in a set. And we want to do that because of efficiency.

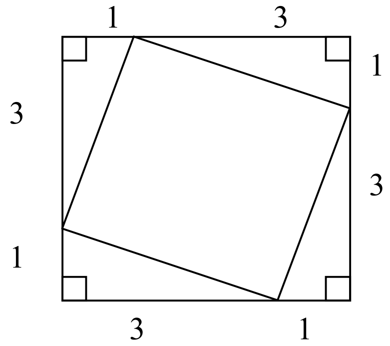
For instance, we might have students solve individual quadratic equations by completing the square. But, if we have completed the square for the general case $ax^2 + bx + c = 0$ in order to develop the Quadratic Formula, there is no need to complete the square to solve any future quadratic equation. We might have students determine the length of the hypotenuse of a right triangle with legs 1 and 3 using an area argument such as in the following diagram, but if we do it in general, we have the Pythagorean Theorem (or the Theorem of Three Squares, as it is called in some of the economies represented here) and there is no need to use the area argument again.

Area of middle (tilted square) = area of large square – 4(area of right triangle)

$= 16 - 4 \bullet 0.5 \bullet 1 \bullet 3$

$= 10$

So the side of square has length $\sqrt{10}$.



The generalization from individual instances to a general formula is a hallmark of mathematics and one of the progressions that students need to see again and again. If we have deduced that $\sqrt{2}$ is irrational, how many other numbers can we prove to be irrational by an analogous argument?

The move from properties of individual figures to properties of all figures in a set can be subtle and unsettling for students. For instance, consider the teacher who wishes to convince her students that the sum of the measures of the angles of a triangle is 180°. But what does this mean? There are subtleties here. Consider statements (1) and (2). They have much the same sentence structure. But there is quite a difference between them.

(1) In ΔABC, $AB + BC + AC = 15$.

(2) In ΔABC, $m\angle A + m\angle B + m\angle C = 180°$.

Statement (1) applies only to certain triangles and is given information in some problems, while statement (2) applies to all triangles and is often isolated as a theorem.

Here is another example of the same type where the statements look even more alike. Each of these statements could be true.

(3) In a triangle, the largest angle is obtuse.

(4) In a triangle, the smallest angle is acute.

Here, statement (3) could be given information about a particular triangle while statement (4) is true for all triangles.

Young children are aware of properties of individual triangles, such as in (1) or (3). But the study of geometry requires that students be able to work with properties of all triangles, such as (2) and (4). These semantic similarities between statements get in the way, and they motivate the use of quantifiers.

(1) It is sometimes true that in ΔABC, $AB + BC + AC = 15$.

(2) It is always true that in ΔABC, $m\angle A + m\angle B + m\angle C = 180°$.

(3) In some triangles, the largest angle is obtuse.

(4) In all triangles, the smallest angle is acute.

How do we know that (2) is always true? Typically, a good teacher gives students an activity: draw a triangle on a sheet of paper, carefully measure its angles, and add the measures.

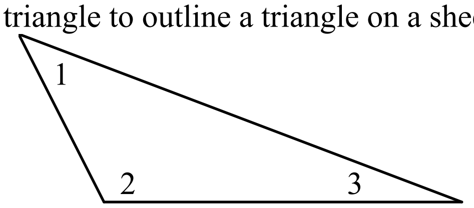
What happens? Though most of the sums students obtain from measuring are near 180°, they are not all exactly 180°. The teacher may explain that measurements are not exact, but some students wonder whether statement (2) is really true always. Maybe the teacher is oversimplifying, just as is done with spelling rules in English such as "i before e except after c", being that there are exceptions such as height and weight, not all of them weird. Maybe the sums of angle measures round to 180°. Maybe the sum is 180° only for triangles within a certain range of shapes. Maybe the average sum is 180°.

There is now a quandary regarding how to proceed because one of the points the teacher wants to make is that you *cannot* make a generalization for infinitely many objects just by looking at specific examples. The activity seems like a perfect hands-on activity but it has failed to help the progression from individual figures to a class of figures. The difficulty is that the strategy used by Miss Smith is fine for asserting the truth of $m\angle A + m\angle B + m\angle C = 180°$ for a *particular* ΔABC, and even for *many particular* triangles but not for *all* triangles.

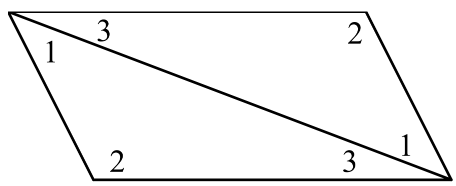
Here is a better activity for making the progression.

Step 1: Cut out a triangle from a sheet of cardboard.

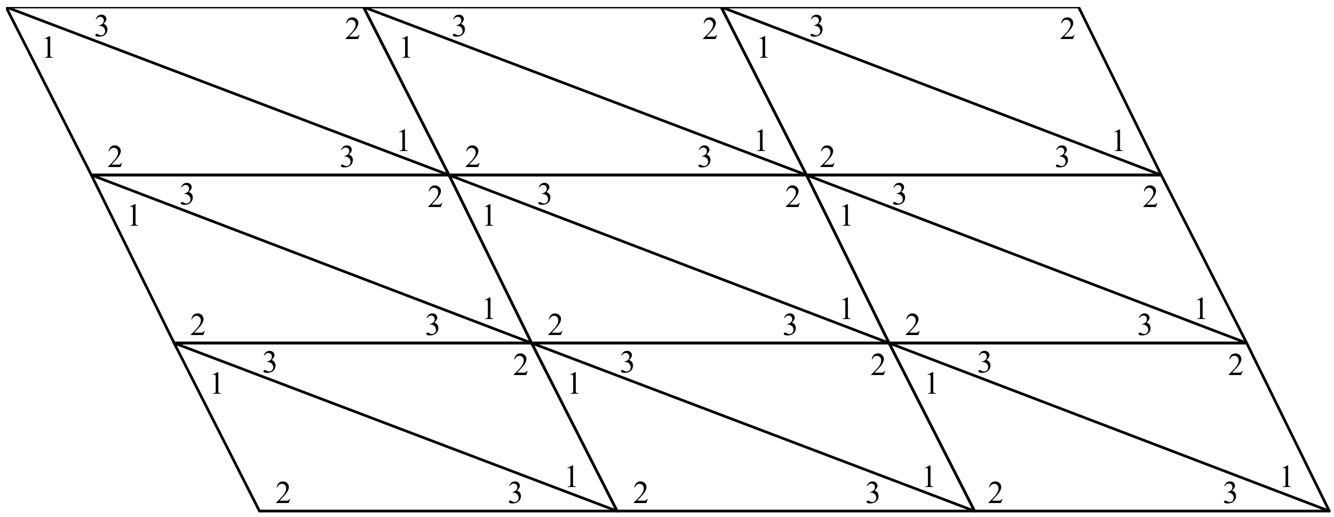
Step 2. Use this triangle to outline a triangle on a sheet of paper.



Step 3. Outline the same triangle again and turn the second triangle so that the two triangles together form a parallelogram.



Step 3: Repeat the parallelogram to tessellate the plane.



When the parallelograms are tessellated, we see that around each vertex are six angles, two copies of each angle of the original triangle. Since the sum of the six angles is 360°, the sum of three different angles is 180°. Now we see that the sum is 180° because it is half of the number 360° that is used for a complete revolution. And we should tell why the Babylonians chose 360°. And we might even tell our students that angle measure does not have to be in degrees. If another unit were used, then the sum would be different. And students should know why this tessellation could not be done on the Earth's surface, not because they would necessarily have that on a test, but to see the significance of parallel lines in this argument and just in case someone asked.

Here a dynamic geometry drawing program such as *Cabri Geometrie* or *Geometer's Sketchpad* can help, because it enables the student to verify large numbers of examples with triangles of all different shapes. But the technology does not suffice; Paul Goldenberg has reported that many students think the computer is pre-programmed to get the results it shows. Deduction is needed.

Progression 4: from inductive arguments to deductive ones and then to deduction within a mathematical system

In the preceding progressions, I have used inductive thinking several times. Inductive thinking is how we live most of our lives. We walk or ride to work using a route that in the past has served us well. We try out something in the classroom and, if it works, we'll use it again and again. Induction is also one of the two mechanisms by which we reason in mathematics. Induction gives us conjectures.

Induction is often misused in school. Children are asked: What is the next number in this sequence? 1, 2, 3, 4, 5, 6, 7, 8, 9, … The correct answer: There is not enough information to tell. It could be 10 if the sequence is that of the counting numbers in increasing order. It could be 11, if the sequence is that of the positive integers not divisible by 10. It could be 0 if the sequence is the sequence of the units digits of the positive integers.

Induction can be subtle. Consider 0^0. It seems reasonable to view 0^0 as the limit of 0^x as the real number x approaches 0. We calculate: $0^2 = 0$; $0^1 = 0$; $0^{0.5} = \sqrt{0} = 0$; $0^{0.1} = \sqrt[10]{0} = 0$; and so on. It seems definitive: $0^0 = 0$.

But it also seems reasonable to view 0^0 as the limit of x^0 as x approaches 0. Now we calculate: $2^0 = 1$; $1^0 = 1$; $(0.5)^0 = 1$; $(0.1)^0 = 1$. It seems just as definitive: $0^0 = 1$. We tried to induce the value of 0^0 and came up with two different values depending on which pattern we wished to follow. So a first step in the progression is to show that induction does not always work even in a closed mathematical setting.

Deduction begins with a single word "if" or "suppose" or "assume", followed by a question "What if?" or "Then what happens?" *Assumptions are important in deduction*. What if the sequence 1, 2, 3, 4, 5, 6, 7, 8, 9, … is the sequence of integers in increasing order that are the days of the months of the year beginning with January? Then we have 1, 2, 3, …, 31, 1, 2, 3, …, 28 (or 29, depending on the year), … It's not the simple sequence we thought!

Deduction is the hallmark of mathematical thinking. We have not taught students the essence of mathematical thought unless they appreciate the power of deduction. The full power, however, only comes when we are aware of the assumptions from which we deduce. Those assumptions are perhaps easier to see in applications, where assumptions become constraints in a problem, than in theory, where assumptions often need to be traced back to a large number of postulates.

You have saved 500 baht. What if you save an additional 150 baht each week? Then what happens? This open-ended question is the essence of mathematical thinking. Too often we tell students what we want them to prove rather than asking them to prove *anything* from the given information and then see how far they can go.

We can do this also with pure mathematics. Divisibilty properties are very suitable for early deduction and many students are curious about them. Suppose m and n are any different integers, each divisible by 7. What can be deduced about m + n? (Some students begin by thinking that m + n is always divisible by 14; deduction shows that m + n is always divisible by 7 and a counterexample shows that m + n is not always divisible by 14.) What can be deduced about mn? What can be deduced about $m^3n^2 + m^2n^3$? Students can discover as well as deduce properties of divisibility.

Deduction often carries us into mathematics at a higher level. The human population of our planet is currently about 6.8 billion and growing at a rate of 1.17% a year according to recent estimates. If we assume those estimates are correct and the growth rate is constant, then what? A first reasonable conclusion is that the population P of the planet is given by $P = 6{,}800{,}000{,}000(1.0117)^n$, where n is the number of years from now.

A nice aspect of this problem is that the question of the domain of n becomes significant to the application and is more than a textbook exercise. Can n be 0? (Yes, that is the population "now".) Can *n* be negative? (Yes, for example, if n = -2, then the formula calculates the population *two years ago* under the assumptions of the problem.) Must *n* be an integer? (No, but then we are forced into asking when exactly "now" is.) This example can be used to give meaning to non-integer and negative exponents.

In this population example, when $n = 1000$, $P \approx 766{,}000{,}000{,}000{,}000$, that is, about 766 trillion people, or over 5,000,000 people per square kilometer of land. In contrast, Macau, the most dense country in the world, has a density of under 19,000 people per square kilometer. What seems to be a weakness of the formula we have used is actually its strength, because we can change the assumptions and so deduce a new formula. We might want to indicate a limit L for the world population; if so, we can obtain a logistic formula for the world population *n* years from now, describable by the recursive formula $P(n+1) = P(n) + .0117P(n)\left(1-\frac{P(n)}{L}\right)$. We may have moved into tertiary mathematics, but we have also given secondary school students a reason for studying that mathematics. In this way, applied mathematics can give at least as much meaning to deduction as pure mathematics.

To understand deduction fully, students also need to see examples where false assumptions lead to nonsense. Perhaps my favorite example of this type is to ask students for any two integers between 0 and 100. Say that the integers are 48 and 61. Then ask for two more, say 9 and 91. Now I assert that I will prove: If 48 = 61, then 9 = 91. One possible proof is as follows:

$$\begin{array}{lrcl}
 & 48 & = & 61 \\
\Rightarrow & 48 \bullet \frac{1}{13} & = & 61 \bullet \frac{1}{13} \\
\Rightarrow & \frac{48}{13} & = & \frac{61}{13} \\
\Rightarrow & 3\frac{9}{13} & = & 4\frac{9}{13} \\
\Rightarrow & 3 & = & 4 \\
\Rightarrow & 3 \bullet 82 & = & 4 \bullet 82 \\
\Rightarrow & 246 & = & 328 \\
\Rightarrow & 246 - 237 & = & 328 - 237 \\
\Rightarrow & 9 & = & 91
\end{array}$$

The point is that even if a person uses valid reasoning, if that reasoning stems from statements that are not true, then you cannot be certain of the truth of any conclusion. But if you use valid reasoning from true statements, then you must get true statements. And, if you reason from a given statement whose truth you do not know, but you get a false statement, then you know that the given statement had to be false. This, of course, is the foundation of indirect proof.

Somewhere before they leave their study of mathematics, students need to be introduced to the wonders of a mathematical system, that is, a system where deduction from a small number of postulates has, over the years, led mathematicians not only already to deduce a myriad of theorems but to be continually proving more theorems. Either Euclidean geometry, or the real number complete ordered field, or a set of postulates for the positive integers are good candidates for such a system. They are good candidates because there are an unlimited number of theorems in each system and so they display the immense power of deduction.

Progression 5: from uses of numbers to uses of operations to modelling with functions

I am a passionate believer that students should see the wonders of pure mathematics, but I am at least as passionate in believing that students must be introduced to the breadth of applications of our subject. If a student leaves a mathematics classroom not knowing why the mathematics is important, it is our fault. We cannot expect teachers of other subjects to tell students why they need to study mathematics. That is our job.

Modeling begins in the primary school. In primary school, we expect that students have seen numbers used as *counts* and as *measures*, with counting units and units of measure, respectively. As mentioned earlier, they should also have seen that, in situations with two directions, positive and negative numbers arise. Fractions and percent show that numbers are also the result of *ratio comparisons*, and such numbers do not have units. π is a wonderful example of an irrational number used as a ratio comparison Numbers also represent *locations*. Street addresses, rank orders, and scales such as the Centigrade scale for temperatures or the decibel scale for sound intensity represent this fourth use of numbers. Numbers also may be used as *identification or codes*, as in charge card numbers or ISBNs. And of course there are numbers simply used as numbers, such as when we examine prime numbers or lucky numbers.

From the uses of numbers develop meanings for the operations. The sum x + y has meaning if x and y are counts or measures, but not necessarily when x and y are ratio comparisons, and almost never if x and y are codes. When x and y are locations such as scale values, the sum x + y does not have much meaning – the sum of two temperatures does not have meaning, for example, yet the difference x – y almost always has a meaning. Each of the operations has fundamental meanings: addition as putting-together or slide; subtraction as take-away or comparison, and special cases of comparison are error and change. Multiplication is area or acting across, size change, or rate factor; division is rate or ratio; powering is growth. The meanings are related to each other just as the operations are: for examples, take-away undoes putting-together; size change undoes ratio.

As a progression, it is fundamentally important that in the primary school these uses of numbers and operations go beyond counts to include non-integers. Then, in the lower secondary school, these uses can be employed to give meaning to algebraic expressions. For instance, in the expression $\frac{y_2 - y_1}{x_2 - x_1}$ for the rate of change between (x_1, y_1) and (x_2, y_2), $y_2 - y_1$ and $x_2 - x_1$ are subtraction comparisons and the division is a rate, so it is no surprise that $\frac{y_2 - y_1}{x_2 - x_1}$ represents rate of change.

From the meanings of algebraic expressions come the situations that functions model. When items with unit costs x, y, and z are purchased in quantities A, B, and C, the sum Ax + By + Cz is an addition putting together rate factor multiplications to arrive at a total price. Linear functions arise from these linear combinations or situations of constant increase or constant decrease. Exponential functions model situations of growth or decay. Quadratic functions model situations of acceleration or deceleration (the rate of a rate), or area. Trigonometric functions model circular motion and are often quite appropriate in situations where phenomena occur periodically. The broad kinds of situations that the various types of functions model should be as much a part of the curriculum as the mathematical properties of these functions, for it is almost certain we would not be studying them were it not for their applications.

It is useful at this point to consider the three levels of modeling: the *exact model*, such as in the number of games necessary for *n* teams to play each other; the *almost-exact theory-based* model, such as in modeling the path of a thrown ball by taking measurements along its path; and the *impressionistic* model, such as when one finds that the population of a region over a particular time interval is described well by a quadratic function and no theory explains that. We sometimes give the incorrect impression that mathematical models are always approximate and messy, but the reality is that many mathematical models are exact. On the other hand, some users of mathematics give the alternate impression – that their impressionistic models are reliable – and we need to caution against that improper inference.

Progression 6: from estimation of a single measurement to statistics for sets of numbers, and from descriptive statistics to inferential statistics

In many quarters, probability and statistics are considered together and separate from other mathematics. I see these two topics for the most part as instances of other progressions. For instance, the calculation of relative frequency is an example of ratio division. The fitting of lines or curves to data is how we model data by functions. Still, there is a progression that is distinctively statistical, namely the consideration of data sets (rather than a single data set) to make inferences about situations of variability, and the role of randomness.

The public often has the view that mathematics is an exact science, and that estimates are never as good as exact values. How wrong this view is! There are often times that estimates are to be preferred over exact values. Consider (1) predictions such as the lifetime of a light bulb or the lifetime of an individual person or the price of chicken next year, or (2) values that are changing constantly such as temperatures or populations, or (3) measurements such as a person’s waist size or score on a memory test, or (4) values that we want to be consistent in a table, such as 3-place decimals in describing the winning percentages of sports teams. In these cases, estimates by convenient nearby numbers, intervals, or distributions are more appropriate models than an exact value. For convenience we may substitute a single number or two to describe an interval or a distribution, numbers we call statistics.

Thus this progression begins with the importance of estimates. Then it moves to consideration of how to describe (estimate) a set of numbers without listing all the numbers. We may use single numbers such as the mean (an example of rate division) or the median (a location), or pairs of numbers as with an interval, or multiple locations such as the 5-number summary (minimum, 1st quartile, median, 2nd quartile, maximum) – five numbers used as locations! We realize that we have lost information in the use of these statistics, so we often return to the full distribution and describe it with such terms as skewness, symmetry, tails, its modes and outliers, and its spread, with statistics such as the standard deviation or mean absolute deviation.

It is often said that statistics is different from mathematics because statistical thinking is probabilistic and inferential, while mathematical thinking is deductive. True, but good statistics uses deduction from hypotheses just as mathematics does. The major difference, in my opinion, is that statistics is applied mathematics in that it arises from data, while mathematics arises from theory. To make this distinction, it is better to use the term *relative frequency distribution* for a distribution based on data, and *probability distribution* for one based on theory, rather than the terms *experimental probability* and *theoretical probability* found in many places. The better terms emphasize that probabilities are always either assumed (often through randomness or from past experience) or calculated from assumed probabilities, whereas relative requencies always arise from data. For instance, in Malaysia in 2007 provisional figures from the U.N. indicate that 235,359 males and 221,084 females were born. Thus the relative frequency of male births was about 0.516. Assuming randomness, the probability that a randomly-selected baby born in Malaysia in 2007 is a boy is , so by making the randomness assumption we can turn the relative frequency into a probability, but far more likely for calculations a person would use 0.516 or 0.52 for the probability.

It is useful to have distributions that arise from data that are not random (such as test scores) and data that are randomly generated from experiments (such as coin-tossing), because we often want to pick a data point at random from a non-random distribution (for instance, if we choose a student at random from a non-random distribution, what is the probability that the test score is greater than some number). This is preparation for the idea of events with low probability, that is, events that are not very likely to happen.

Although it is not uncommon to separate statistics from other mathematics, there are several advantages to teaching them together. First, statistics requires dealing with expressions involving absolute value, square roots, binomials, and other algebraic language. Second, distributions are functions and can be used to strengthen function concepts such as end behavior, symmetry, and limits. Third, distributions can be modeled by functions, such as when linear regression is used to determine the line of best fit for a set of data. Fourth, transformations that are applied to data such as scaling and translating in order to normalize the data are often also applied to functions in order to study their behavior.

Once students have dealt with data and they, students are ready for hypothesis testing and inferential statistics. For instance, one might ask about the Malaysian 2007 births, could a ratio this far from 50% males and 50% females have occurred by chance? In other words, *if* (i.e., *hypothesizing that*) the sex of a baby is random between males and females, what is the probability that 235,359 males would be born out of 456,443 births. In this way, null hypotheses and alternate hypotheses are assumptions made for a particular situation and thus give another opportunity for deduction. The only difference is that the answers to questions of inference are typically probabilities ("The probability is x that data like these would arise if the data were random."). In this case, the probability is very, very small that such numbers of males and females would occur. With small numbers, we would calculate a probability like this using binomial coefficients; with large numbers such as these, we use a normal distribution. But calculating is not the only way. Students should learn simulation such as the use of Monte Carlo techniques. Among all the direct statistical tests, I think it is easiest to begin hypothesis testing with Chi-square tests. Finally, a major goal of teaching this content should be to immerse students in examples of how statistics can be used to gain valuable information about and make inferences from data in order to combat the common societal view that statistics are not to be trusted.

Progression 7: from the idea of same size and shape (same shape) to a general definition of congruence (and similarity) applying to all figures, to conditions for the congruence (and similarity) of simple geometric figures, to the application to all figures and graphs and the Graph Transformation Theorems

The treatments of congruence and similarity in K-12 schooling do not typically follow a smooth path. Congruence in lower grades is "same size, same shape", applying to all figures, yet when a concerted study is begun in later grades, the figures are often restricted to be triangles and perhaps circles. Later, perhaps only in college, a definition of congruence in terms of transformations is provided that brings the student back to consideration of all figures. In my opinion, this is not the best progression. The restriction of congruence to simple figures is not at all helpful for student understanding of the idea.

Throughout schooling it is possible to consider figures as congruent if and only if one can be mapped onto the other by a composite of reflections, rotations, translations, and glide reflections (or any one of many other equivalent definitions). This gives an intuitive picture that can be reinforced by reference to products produced by a machine, tessellations, duplicate copies of a photograph, etc.

There are two advantages to this sequence. By considering the graph of a (typical elementary) function as a set of points in the plane, the idea of congruence easily extends to the congruence of graphs. A particular special case are graphs that are translation images of each other. Equations for these graphs can be found using the Graph Translation Theorem: In a relation described by a sentence in x and y, the following two processes yield the same graph:
(1) replacing x by $x–h$ and y by $y-k$; and
(2) applying the translation $T(x,y) = (x+h, y+k)$ to the graph of the original relation.
It is somewhat surprising that this theorem is not found in many of today's books, given the number of corollaries that are found. Call the original relation the parent and its translation images the offspring of the relation. Then the following correspond:

Shape of graph	**Parent**	**Offspring (image)**	
Line	$y = mx$	$y – y_0 = m(x – x_0)$	Poinr-Slope
Line	$y = mx$	$y – b = mx$	Slope-Intercept
Circle	$x^2 + y^2 = r^2$	$(x – h)^2 + (y – k)^2 = r^2$	
Parabola	$y = ax^2$	$y – k = a(x – h)^2$	
Sine Wave	$y = \sin x$	$y = \sin(x – c)$	Phase Shift

Parabola intercepts	$ax^2 = c \Leftrightarrow x = \pm\sqrt{\frac{c}{a}}$	$a(x-h)^2 = c \Leftrightarrow x = h \pm \sqrt{\frac{c}{a}}$ Quadratic Formula
Exponential	$y = b^x$	$y = ab^x$
Logarithmic	$y = \log_b(x)$	$y = \log_b(ax)$

Similarity is often restricted in K-12 schooling to polygons and polyhedra. This, too, is unfortunate, as so many interesting instances of similarity involve more complex figures. Every day students see pictures on television and other media that are similar to the actual objects being pictured. In the upper primary and lower secondary school, scale drawings, physical models of large objects, and maps can be used to demonstrate similarity. Dilatations (size changes) can be introduced to create larger and smaller images of given figures. A definition of similar figures in terms of transformations is easily given once there has been the corresponding definition for congruent figures. The Fundamental Theorem of Similarity, that in similar figures with ratio of similitude k, corresponding angles have equal measures, corresponding lengths are in the ratio k, corresponding areas are in the ratio k^2, and corresponding volumes are in the ratio k^3 can be applied to the question of the existence of giants such as those students have read about in fairy tales and see in cartoons. Under a general definition of similarity, all parabolas are similar and so are all graphs of exponential and logarithmic functions.

These properties of the graphs of functions follow from the Graph Scale Change Theorem: Translation Theorem: In a relation described by a sentence in x and y, the following two processes yield the same graph:

(1) replacing x by $\frac{x}{a}$ and y by $\frac{y}{b}$; and

(2) applying the scale change $T(x,y) = (ax, by)$ to the graph of the original relation.

As is the case with the Graph Translation Theorem, this is a powerful theorem with many useful corollaries that are important precursors for the study of integrals in calculus and assist in the understanding of graphs of all functions.

Shape of graph	**Parent**	**Offspring (image)**
Circle	$x^2 + y^2 = 1$	Ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$
Hyperbola	$xy = 1$	*Hyperbola: $xy = k$
Parabola	$y = x^2$	*Parabola: $y = ax^2$
Line	$y = x$	*Line: $y = mx$
Sine Wave	$y = \sin x$	Amplitude/Period: $y = A\sin(Bx)$
Exponential	$y = b^x$	*Change of Base: $y = ac^x$
Logarithmic	$y = \log_b(x)$	*Change o Basey = $\log_c(x)$

*Image is geometrically similar to the parent.

Progression 8: from scientific calculators to graphing calculators to computer algebra systems

In many economies the major role played by mathematics beyond arithmetic is – whether intentional or incidental – as a sorter; that is, to separate out better students from poorer ones regardless of their interests or vocational goals. In years past, using mathematics as a sorter was defensible because not many people would be helped by knowing mathematics beyond arithmetic and simple geometry and because the examination questions involved skills that were needed by the small percent of the population who needed higher mathematics. However, today we feel that most people would be helped by knowing about the behavior of functions, the fundamentals of descriptive and inferential statistics, and many other mathematical topics not found in the primary school curriculum. And today there exist hand-held devices and computer software that can accomplish any of the difficult calculations that often served to sort students in the past. These tools make it possible for the first time to realize the goal of significant mathematical literacy for all. A corollary to this argument is that if one does not allow this technology, then the sorting done by mathematics is often due to performance on tasks that can be carried out automatically by a machine and not on a person's ability to understand and apply mathematical ideas.

Most of the individuals charged with the task of creating standards in our economies today were in school when there were no hand-held calculators. These individuals, almost all of whom were very successful in their mathematics study, often dismiss today's calculator and computer technology not only as unnecessary but even as harmful to the mathematics education of today's youth. I profoundly disagree.

Since 1975 we at Chicago have been developing curricula using the latest technology – first, just scientific calculators, then graphing calculators, and most recently, computer algebra systems (CAS). Our experiences have convinced me and those who work with me that this technology enhances both the conceptual understanding and problem-solving ability of students. For slower students, those who in the past might have been sorted out of mathematics and even out of advanced schooling because of difficulties with our subject, the technology is particularly important. It enables them to understand mathematics that would otherwise defeat them. It is life support, helping them to survive their mathematics courses, or it is a crutch, helping them until they can walk. For better students, the technology is an *extender*, helping them to move more easily into more advanced mathematics. And for all students, the technology sends a message – that mathematics is current and relevant in today's world.

Today's advanced technology is so sophisticated that it, like any other advanced concept, can overwhelm students who have not had experience with the corresponding work that is not so advanced. The order is straightforward: from early primary school, students should be working with calculators. In later primary school, scientific calculators that can deal with fractions should be used. In early secondary school, students should begin working with calculators that enable graphing and geometry. And, in later secondary school, students should have calculators with computer algebra system capability. The CAS technology, the newest in the arsenal of technology that can do mathematics, we have found to be particularly important for students who have had trouble learning algebra. For the first time, they can play with algebraic patterns with confidence that what they are doing will lead to correct answers, and by seeing the patterns in the answers, they gain proficiency.

These practical reasons for a technology sequence have a theoretical counterpart in the well-known paper-and-pencil algorithms that traditionally dominate the learning of arithmetic and algebra. Paper-and-pencil is a technology whose applicability hundreds of years ago, when paper was scarce and pens required ink, was strikingly analogous to the situation today in that the more affluent people and societies have access while the poorer lag. Schools could not begin to teach everyone the paper-and-pencil algorithms until almost all students had their own paper and ink supplies, and it is still the case in some schools that these are scarce commodities and rationed.

The procedures employed to obtain answers in arithmetic and algebra are carefully sequenced in today's books. We can also expect the analogous procedures to obtain answers using calculators to be carefully sequenced; the only problem is that many new and more powerful calculators are getting on the market each year.

To those who believe that paper-and-pencil work *is* mathematics, and calculator work is not, we might note that large numbers of students in the United States blindly apply paper-and-pencil algorithms with no idea of why they work or whether their answers make sense. They cannot multiply $\frac{3}{4}$ by 8 unless they change the 8 to $\frac{8}{1}$. To multiply 357 by 8000, they first put down three rows of zeros (not just three zeros). They are totally at a loss to explain long division. Algebra is even less understood. Polynomials are factored with no idea that if a number is substituted for the variable, the value of the original and factored expression will be the same. Equations are solved with no idea why one would ever want to solve an equation. Rational expressions are operated on with no idea of how to check whether the answer is correct except to look in the back of the book and hope that an answer is there.

Having the technology does not automatically eliminate these deficiencies, but it enables both student and teacher to spend time on the important ideas and not lose the forest through the trees. In almost all situations, paper-and-pencil manipulation should be a means to an end, not the end itself.

But it is certainly the case that some students overuse calculators, just as many o us use paper and pencil to calculate answers that we should have memorized. In our experience, this is particularly true of students who did *not* have calculators while they were learning the algorithms. Such students see calculators merely as a time-saver and do not understand their use in helping to learn facts and algorithms and to check work. Students who have calculators while they are learning mental and paper-and-pencil arithmetic are forced from the start to make decisions about when it is appropriate to use any or all of these means and seem to be able to make wiser decision later about the use of any of these technologies.

Progression 9: from a view of mathematics as a set of memorized facts to seeing mathematics as interrelated ideas accessible through a variety of means.

The sequence of ideas in mathematics can proceed logically as it does in many economies with the teaching of geometry and in many college courses. It can proceed algorithmically, that is, by the complexity of the algorithms, as it has traditionally done in our teaching of arithmetic, algebra, and calculus. Some have tried courses in which the mathematics of a topic proceeds in the order of the historical development of the topic. In the previous eight progressions, the mathematics proceeds from the cognitively simple to the cognitively more complex. This is the vertical dimension of school mathematics, from bottom to top, from lower grades to higher grades.

But there has to be a horizontal dimension, in which the algorithmic order, the logical order, the mathematical modeling, and the representations all play roles in the student's learning of the mathematics being studied. To know *how* to do some mathematics without knowing *why* you would want to do it, or *why it works*, or how to know you are right is insufficient. Cognitive scientists tell us that being able to connect and categorize helps learning. They also view representation and metaphor as the ultimate tests of whether someone understands a particular idea. Students need to be able to check their work by appealing to logic, or an application, or a representation. In the UCSMP curriculum, we call this the SPUR approach to understanding – **S**kills, **P**roperties, **U**ses, and **R**epresentations.

Consider for example the concept of absolute value. **S**kills associated with this concept: calculating $|x|$ for any value of x; solving sentences such as $|x| = k$, $|x| > k$, $|x| < k$, $|x - a| = k$, $|ax + b| < c$, $|x^2 + bx| = 10x$, and so on, of increasing complexity.

Properties associated with absolute value include the definition:
$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$; $x \leq |x|$ for all x; $|xy| = |x||y|$ for all x and y; $|x| + |y| \geq |x + y|$, and so on.

Uses associated with absolute value come from the idea of *distance*, namely, $|x|$ is the distance from x to 0 on the number line; $|x - y|$ is the distance from x to y . Special cases of distance are (undirected) change; error; comparison. For instance, in manufacturing an object when the error between the desired length L of the object and the actual length is A, the error is $|L - A|$, and if lengths are measured in millimeters and we wish that error to be less than 0.1 mm, then the object's length must satisfy $|x - A| < 0.1$

Representations of absolute value are on the number line or coordinate plane. or instance, the solutions to the inequality $|x - A| < 0.1$ are all points within 0.1 of A on a number line, or we can graph $y = |x - A|$ in the coordinate plane and look for the values of x corresponding to those values of y that are within 0.1 of the x-axis.

The SPUR dimensions of understanding of a general concept can be applied to specific situations as well and can illustrate interrelationships among the first eight progressions. Consider this cartoon that appeared in a daily newspaper in the United States.

The numbers 9.9 and 9.2 here are statistics; in fact, they are means. To us the cartoon is humorous, finishing with a common joke when a mean of counts is not a whole number. But to many people, the 9.9 and 9.2 indicate the lack of reality of mathematics. Just as you cannot have a "point-two" friend, you cannot trust statistics. And if students have not made the transition from whole number to rational number, and if they have not dealt with statistics, they will have great difficulty getting this joke.

The understanding that seems to be lacking here is the notion that in the process of gaining simplicity by using a single number to describe a set of numbers, something is always lost. Here we have single numbers describing entire distributions, and we have lost the distributions.

The next day the cartoonist continued this theme.

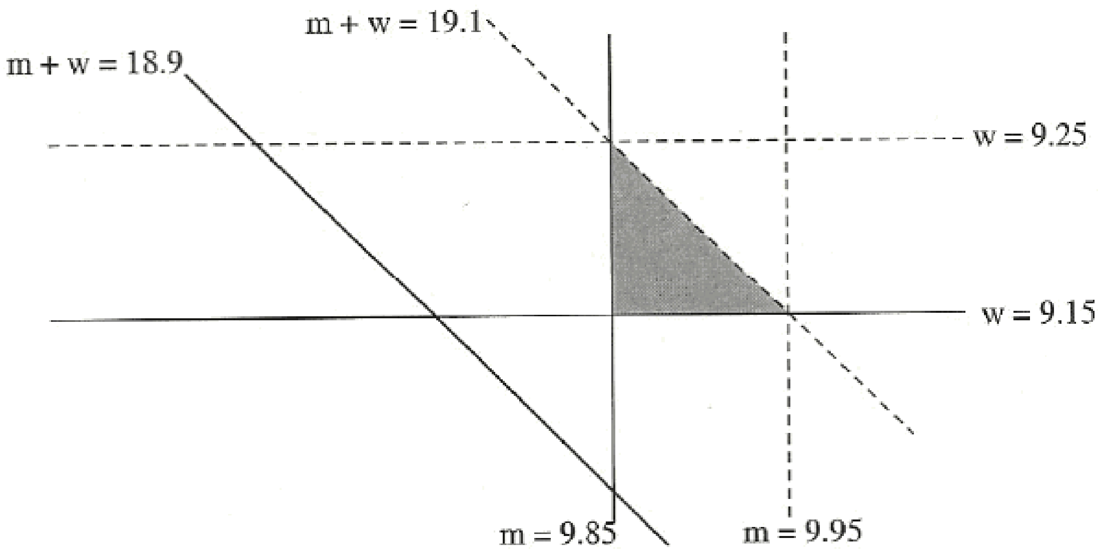
When I saw this cartoon, I became intrigued. Assuming the same number of men and women participated in the poll, how can you get 9.5 as an average of 9.9 and 9.2? Thus begins a mathematical analysis of the situation.

The numbers 9.9, 9.2, and 9.5 are all rounded to the nearest tenth. Each number stands for an interval. If m and w are the values for the average number of close friends a man and a woman have, then **$9.85 \le m < 9.95$** and **$9.15 \le w < 9.25$**. (I am assuming we are rounding up all decimals that end in 5.) Within these intervals we wish to know whether it is possible to have $9.45 \le \frac{m+w}{2} < 9.55$, or, equivalently, **$18.9 \le m + w < 19.1$** .

So one way to answer the question is to give pairs of values of m and w that satisfy the three inequalities written above in bold.

To find some pairs is not particularly difficult, but it seems like a very difficult problem to find all possible values. But if we examine the graphical representation, algebra, geometry, and statistics come together in a beautiful way. The graph of $9.85 \leq m < 9.95$ is a vertical stripe; the graph of $9.15 \leq w < 9.25$ is a horizontal stripe, and the graph of

$18.9 \leq m + w < 19.1$ is an oblique stripe between the lines $m + w = 18.9$ and $m + w = 19.1$. And all stripes contain their lower boundaries but not their upper boundaries, as shown on the next page.



The values of m and w that satisfy all three inequalities provide ordered pairs (m,w) that are either on or in a triangle. For instance, one pair of values is 9.88 for m, 9.17 for w, and so m+w = 19.05. So it could have been that the men in the study had, on average, 9.88 friends and the women 9.17 friends. They would have an average of 9.525 friends. And when these numbers are rounded, we get 9.9 friends for men, 9.2 friends for women, and 9.5 for the entire group just as we wanted.

This example involves algebraic skills (the solving of a system of inequalities), properties (realizing the meaning of a measurement to a single decimal place as well as the principles underlying the transformation of the inequalities into nice form), uses (the modeling of friendship by a rational number), and representations (the graph and the geometric representation of the algebra).

This example is cute but was not picked because it is cute. The horizontal integration of mathematics is as important as its vertical progression; otherwise students naturally believe that if they can answer questions involving dozens of separate mathematical ideas, then they have learned mathematics well. But they have not learned one of the most messages of mathematics: that the mathematics they are studying operates within a single logical system that ranges from everyday arithmetic through the most complicated of functions that one studies in analysis and includes measurement, algebra, geometry, trigonometry, probability, and statistics along the way.

It is possible to extend each of the progressions described here to the mathematics that students may encounter in college. Linear and exponential models pave the way for the consideration of logistic models. The transformations basic to Euclidean geometry lay the foundation for affine and non-linear transformations. Relationships within and between algebra and geometry exemplify the morphisms found in higher algebra. And it has been impossible to cover all the bases of mathematics in grades 7-12. The study of combinatorics, probability, and limits, and the logic of definition and of propositions came to mind as I wrote this essay and I am certain there are other important topics I have missed. I have also not dealt with nurturing the affective dimension of schooling – that, whatever we do, we have not succeeded with an individual student unless that student views mathematics without fear, with the desire to learn more, and with the awe that our subject deserves.

Learning Progressions and Standards

The essay above is an expansion of the paper I wrote for the APEC conference, a paper whose title was given to me. At the conference, the paper was presented in the session dealing with "standards" and I was asked by the organizers also to add some comments in this regard.

We recognize that standards can play a variety of roles in mathematics education, both in curriculum and evaluation. (1) Standards may determine curriculum, forcing all materials in a particular economy or geographic area to adhere to them. (2) Standards may guide curriculum, serving as suggestions to which an ideal curriculum might aspire. (3) Standards may represent criteria for minimal performance in order to move to a higher level. (4) Standards may set goals for high performance at a given level. Typically, evaluation standards are more explicit than curriculum standards, though sometimes the same standards are used to determine both curriculum and evaluation.

It is not uncommon to see standards conceptualized as a two-dimensional matrix, in which one dimension consists of strands or areas of mathematics and a second dimension is grade levels. Thought of in that way, five progressions in this paper lie as follows:

Progression 1:	Number
Progression 2:	Algebra
Progression 3:	Measurement
Progression 6:	Statistics
Progression 7:	Geometry

Three other progressions are in the realms of process standards, i.e., they cross all content areas. They are related to three of the four SPUR dimensions of understanding described in Progression 9.

Progression 4:	Reasoning (Properties)
Progression 5:	Modeling (Uses)
Progression 8:	Algorithmic Thinking (Skills)

The progressions mentioned at the start of this paper as being so common they not need be explicated here are those of deduction and of algorithms, and are related to Progressions 4 and 8 within this paper. The horizontal Progression 9 can be viewed as an integrative progression tying together the other eight areas and in which the fourth SPUR dimension, Representations, plays a major role.

Although the learning progressions described in this paper are, for the most part, directed at the secondary (grades 7-12) level, I have purposely not tried to be more specific and identify a particular year or years for a particular aspect of a progression. Although it is often useful, both in theory and in practice, to treat students of the same age as if they are cognitively alike, they do differ, and some are ready for particular ideas earlier than others. This readiness depends to a great extent on the expectations set in earlier years both in the home and in school, on the time that a student has in which to devote to mathematics, on the interest that the student has or displays in learning mathematics, and, in some economies, on the ability of the home to provide support for student learning. These criteria for readiness are more significant at the secondary level than at the primary level and they account for the fact that in many economies, the mathematical requirements for students begin to be differentiated at the secondary level on the basis of student performance and/or interest. Virtually all economies have realized that, at the secondary level, a “one size fits all” set of standards is not workable and, at some time, there needs to be what mathematics is for all students and what mathematics is for those who express more interest, desire, or ability.

There are very few observations that can be made about the learning of mathematics that apply in all cultures, but one of them, known from the very first international studies of the 1960s, is that *time-on-task* is a significant variable in performance. In the high-performing economies on international comparisons of mathematics performance (Singapore; Korea; Japan and Hong Kong, China), students put in large amounts of time outside the classroom, often in organized tutoring centers (sometimes called “tuition” or “juku”) or with individual tutors. These same conditions, not as well-organized, exist in high-performance public and private schools in the United States. In economies where vast numbers of students do not attend secondary school, or must work or do chores many hours a week in addition to their schooling, or have little or no access to technology, students cannot be expected to proceed through these learning progressions as quickly.

Further arguing against identifying particular grade levels for aspects of a particular learning progression is the ever-changing world of mathematics itself. The learning progressions related to the paper-and-pencil solving of equations in algebra are challenged by technology that can solve the most difficult of these equations in the same way that it solves the easiest. Statistics, which a half-century ago was rarely mentioned in standards, is now viewed in many economies as an important area of secondary mathematics, taking time that used to be devoted to other mathematical areas. As recent as twenty-five years ago, mathematical modeling was viewed in most economies as a tertiary area. Dynamic geometry technology has changed the ways in which we view geometrical objects.

The lack of specificity in the progressions is also due to my view that there are innumerable ways to approach mathematics meaningfully. In this paper I have suggested some ways that I hope will provoke others to examine the mathematics represented in their standards and the materials used in their economies and organize learning progressions that are suitable for the students in their economies.

How to Encourage Students Learning Mathematics Themselves: "Double Class—Real and Virtual Class"

Shangzhi Wang, Capital Normal University

Fengjuan Hu, Capital Normal University

February 2010

Summary

1. Background

At present, curriculum reform in Basic Education in China is at a stage where further explorations are needed. While curriculum standards require students to cultivate their abilities in autonomous learning, cooperative learning and exploratory learning, many problems exist in teaching practices and the solutions to these problems are usually superficial. The teachers and students devote efforts and cherish every second to work them out, but their ultimate goal is just earning a higher score and the method is simply doing more homework. Although the behavior of pursuing higher scores is admirable, we still need another consideration if it is essential for the students to devote so much time on their homework.

Mathematics education of China faces great challenges. They are how to encourage students learning mathematics themselves, to help the students to boost their interests in mathematics and to raise their confidence and ability to learn. Hence we start to think of new ways.

2. Two Successful Cases

2.1 R. Moore Teaching Method

R. Moore was a very excellent mathematician and was especially a very successful mathematics educational scientist. He had cultivated excellent mathematicians in U.S.A and had offered a successful teaching method in mathematics education, particularly "problem course", which has influenced many mathematics educational scientists and mathematicians. We think that the core idea of the Moore teaching method is to encourage students to learn mathematics themselves and during this learning process the teacher will give students effective directing. This teaching and learning method will not only help students obtain some mathematics content and results, but it is very important to improve the student's learning ability and to increase the student's learning confidence and interest. We like to use Moore's educational idea to improve our work.

2.2 Mathematical modeling Activity of China

The spread and application of a network in distance-education has provided us with some new ideas. On the other hand, the exploration of mathematical-modeling has a history of more than 20 years in secondary education, and educators in this field have abundant experience and resources. In different cases, students experimented on mosquito-coil redesigning, the length of shoelaces, the problem of polyhedron-folding, chessboard redesigning, post-service optimization, the problem of packaging, shelter-forest problems and so on. Now we have an accumulation of more than 12,000 papers delivered by the students, some of them are recommended to attend college as a result. Meanwhile, the teachers gained a lot of experience during these practices. In the process of teaching and learning mathematical-modeling, the teachers and students are not only involved in traditional teaching, but they also have to search for different materials through all kinds of ways, and they use internet as their main tool to interact.

Mathematical modeling activity impels some changes in mathematics education. For example, mathematical modeling has already entered into the classroom in high schools; Modeling has changed "teaching" and "learning"; Modeling cultivates the awareness about asking questions and innovating spirit; Modeling has brought great changes to mathematics curriculum in China.

Conventional mathematical modeling can be divided into the following steps:

Teachers provide students with the background of problems, students collect information and identify the issue

To determine the outline of the investigation or information according to the problem,

To search on the Internet or do the appropriate investigation during their spare time.

To solve the problem independently, seek appropriate help and guidance if necessary;

To discuss and exchange results, teachers and students appreciate, question and evaluate each other.

In this process, first of all, we encourage individuals to study and think independently. On this basis, students can form a mathematical modeling group, and discuss together, inspire mutually, divide the task, and explore for the solutions.

Students like to do this activity very much. Here are some titles of students' papers:

“ Optimization problems of banking counter" Chenle's group

• “To determine the appraisal index of basketball shoes" Zhang Boyang's group

• “Quantitative analysis of the value of stocks" Deng Xiaoran's group

• “Research on vehicle emissions" Liu Gezi's group

• “Optimization of computer keyboard" Zhu Chenran's group

• “Public Transport in Beijing" Wang Zihao's group

• “Mathematical problem in rope skipping" Meng Han's group

• “ Functional relation between the time and date of raising flag in Tiananmen Square”,Li Shuo's group

“The changing tendency of death toll reported in an earthquake” Gao Yinxiang's group

• "Forecast of arrival time of population peak in China" Ye Mai's group

• "The functional relation and determinants of applied frequency of Chinese characters" Chen Chongyao's group

• "Relationship between profits and cost of clothes, original cost, number of pieces, total profit, increased quantity of sale for cutting prices, etc." Shen Da's group

• "Charges of Telecom mobile phone's new package" Ludi's group

• "The relationship between basketball shooting angles and hit rate" Song Chang's group
• "Gold futures" Wang Kaizheng's group
• “Research the price changes of mobile phone with a function" Yueguang's group
• "Trend in New Year box office," Lu Yi's group
• "Relationship between the rotation of bicycle pedal and the distance traveled" Chen Yingjiao's group
• "Calculating bank rate" Cao Zhengwang's group
• "Out of the strange cycle" Lv Lansong's group
• "Throwing solid ball" Zhang Yubai's group
• "The best laundry program" Le Shuo's Group
• "How to shoot to enhance the hit rate" Gong Zibo's group
• "Optimal design of beverage cans," Chen Zhaochu's group
• "The relationship between the area of shadow cast by the south window and the time of the day," Wang Xueyun's group
• "Which is more suitable for you, Shenzhouxing or M-Zone?" Ye Shiqing's group
• "The principle of diminishing marginal utility in Running” Shen Sicheng's group
• "The best initial speed for vehicle's sideslip" Guo Hongtao's group
• "Aircraft bomb problem" Liliang's group
• "The number of light, angle, wattage, distance and the area illuminated" Chengnan's Group
• "Changes in Watermelon's price and forecast of future prices" Wang Qingnan's group
• "The functional relation between temperature of water in a bouilloire and the electricity consumed" Ni Zengtao's group
• "Flickering flame of fireworks in the sky and the weight of flame granule" Shen Yichen's group
• "Mathematical analysis in the Billiards" Liu Yehong's group
• "The relationship between the temperature of the asphalt road and the temperature of air near the ground" Tan Wangshu's group
• "Research on water temperature change" Wang Zhongshu's group
• "Idea from ‘ink spreading’" Wang Shu-yu group
• “The relation between the rotation angle of gas stove knob and gas volume used" Yang Liqiang's group

We hope to use these experiences to make a new teaching and learning pattern to encourage students to learn mathematics themselves.

3. Project and Challenges

In 2007, the experiment of curriculum reform in ordinary senior high school in Beijing is to be launched. The experiment, which is led by the Civil Basic Education Reform Commitee, consist of 16 major research projects, all of which shall be delivered and executed individually. One of the major research projects is the "Deepening the Reform of Teaching Methods and the Scientific Applications of Information Technology” project, whose characteristic is to combine the virtual classroom online and the real classes (which is also called dual class combination module or dual class for convenience). The project is directed by the Beijing Education Association, and subjects like Chinese, history, geography, music and math are involved.

In the 1st round of experiment, subjects as Chinese, history, geography and music are involved, and routine teaching in virtual classes are combined with conventional teaching methods. The students' interests and confidence can be lifted in this way, while their abilities to question, to communicate, and to express can be improved, and their learning concept be changed dramatically. The discipline of mathematics is characterized by abstractness, logicality and its wide applicability. The students have little incentive in a conventional classroom, so many teachers wonder if they could learn autonomously in virtual classes. With much doubt we began this experiment on mathematics.

With its own characteristics, the mathematic discipline is facing tremendous pressure and challenge:

Is there a certain way that mathematics can be worked out? Can the students learn mathematics via the Internet with the help of the teachers?

What kinds of content are suitable for autonomous learning?

Is there a stable teaching model that can be spread?

4. Objectives and Thoughts

Faced with problems in curriculum reform, we laid down the objectives for the experiment.

1. To promote autonomy, raise their interest and confidence.
2. To explore a new mode that will help encourage autonomy using network platforms.
3. To build and encourage the share of superior resources.

After discussions between experts, we make our thoughts clear and it can be indicated as follows:

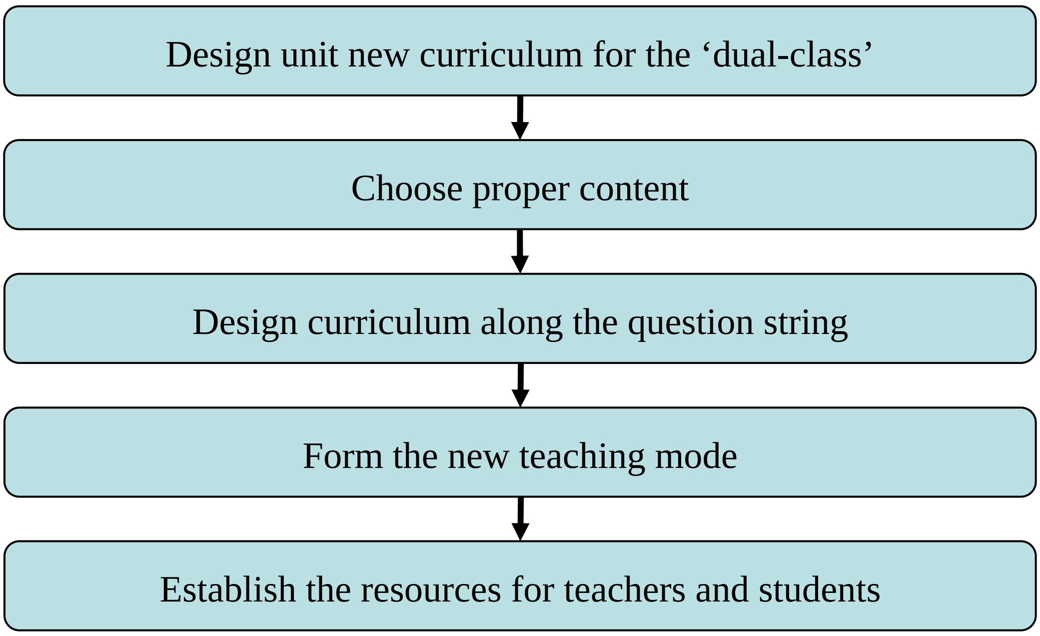


Figure 1: the train of thought for experiment

1. Design unit new curriculum for the ‘dual-class’.

   The so called unit new curriculum refers to one chapter or several chapters. It can be a complete math activity or a special optional topic.

At present, text books used by students are actually written for teachers not for them, making it hard for students to read, and thus their requirements and needs can’t be satisfied. As a result, with the design philosophy “macro-curriculum” we try our best to make learning, teaching, evaluating, and professional development improve in parallel.

2. Choose proper content. Here are the rules for choosing:
   1) Mathematics content should be characterized by:
      - **Analogy**: From plane vector to space vector, from sine function to cosine function, form parithmetic progression to geometric progression. Conic curve.
      - **Vision:** Learn the definition and attributes of trigonometric function as well as identity transformations by virtue of unit circle. Solid geometry introduction is also included.
      - **Activity**: We offer a wide range of statistical problems to choose from. Students participate with enthusiasm, experiencing the whole process of cooperation.
      - **Summary:** Algorithm review knowledge by virtue of algorithm. (Solve inequality, dichotomy)
      - **Reaso**n **and argumentation:** Review the important theorems and definitions we have learned in the past by studying this new part.
   2) The content we choose must be complete.

3. Design curriculum along the question string.

The big challenge is how to arrange the content we have chosen to appeal to students. Here we try our best to make some innovation. We borrow the trick in America and continue the question list.

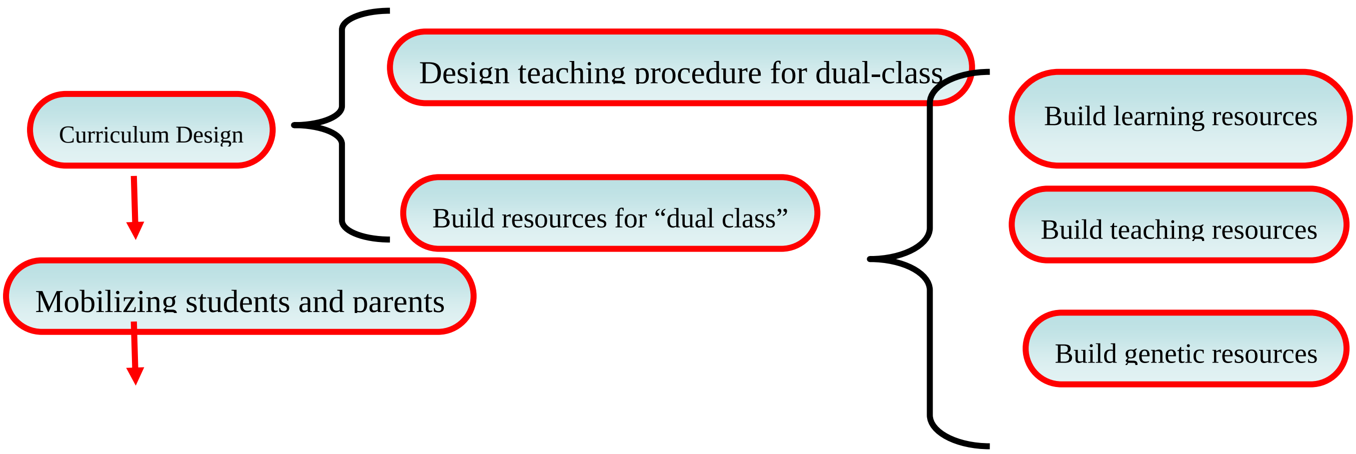
4. Form the new teaching mode

In the virtual class, most of the time, students learn by themselves, collaborate and discuss. Teachers just provide some interesting related applications and questions. This contrasts with the real class, where teachers are the leading role. They interpret definitions, explain main methods and guide students to discuss and solve the trouble happened in the virtual class.

5. Establish the resources for teachers and students

In order to make our new teaching mode “Double Class-Real and Virtual Class” perfect, it is essential to develop and accumulate the lively and vivid resources which are so significant for both students and teachers.

4. Teaching mode in “Double Class-Real and Virtual Class”



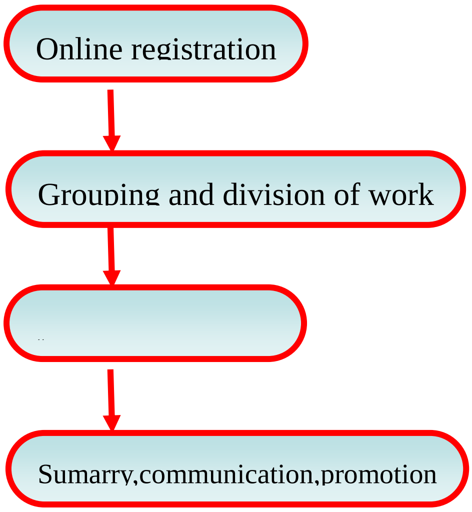


Figure 2: teaching procedure for dual-class

Our teaching mode is fixed gradually after many experiments. Figure 2 has shown all the steps which are all crucial points.

1) Curriculum Design

According to the characteristic of dual-class, the design and accumulation of learning resources and teaching resources should be based on the question string.

2) Mobilizing students and parents

For those ignorant of this new mode, some students may avoid participating in case it affects their scores. So the mobilization is necessary. Dual-class is based on the net, some students can't control themselves well, and as a result, we should chat with teachers to gain their support and trust.

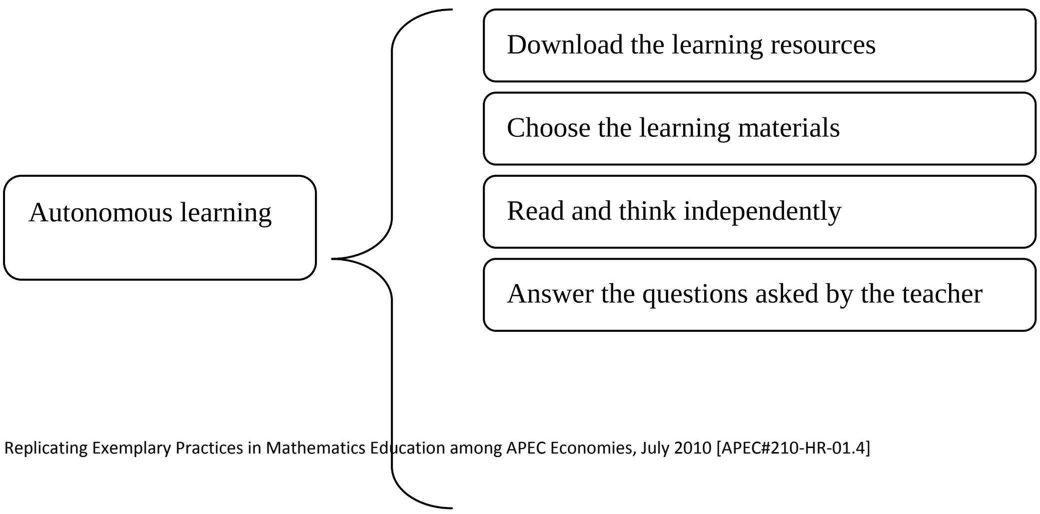
3) Online registration

Entering "virtual class", the precondition is registering online, through which teachers can supervise, manage or record everyone's status of learning. Students and teachers can discuss or communicate together freely on this platform.

4) Grouping and division of work

With their own definite task, students can easily engage respectively on their own initiative. This can cultivate their ability in collaborative learning as well.

5) Decide whether to use the "Double Class-Real and Virtual Class" according to the content.



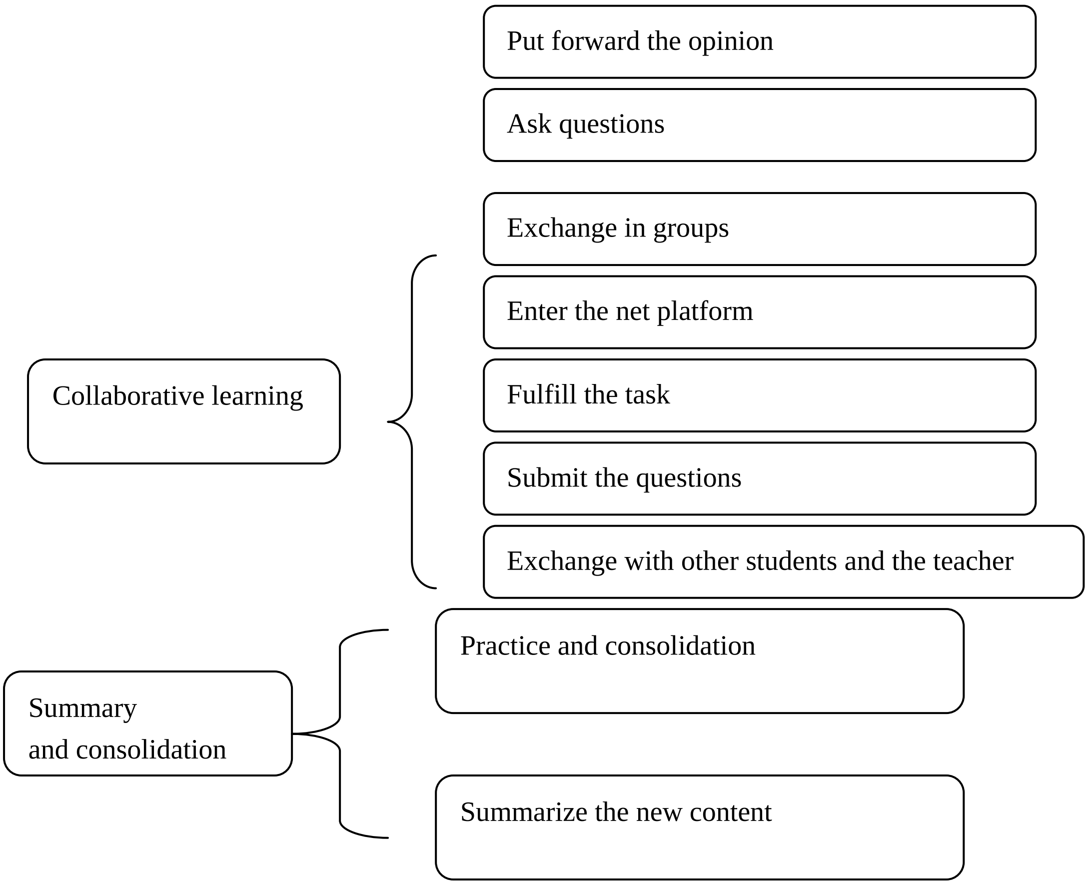


Figure3: Teaching procedure for “virtual class”

Figure 3 shows the teaching procedure in an ordinary situation. It can be adjusted if necessary according to the specific condition. It is clear that leaving sufficient time and space for students can arouse their interests in math greatly and raise their confidence. They dare to question and argue; they learn how to communicate and cooperate.

Here are some situations in the “real class”

- In the initial period, teachers let students get a general idea of what they will learn.
- Teachers explain the questions most students meet in the virtual class
- Teachers remark on the reports that students present in the real class and help them to think deeply.

5. Overview of our experiment

More than 20 schools have signed the cooperation agreement to engage in our experiment since the project has been launched by Beijing Municipal Education Commission and Beijing education society.

Table1: Schools that participated in the experiment and content involved

Number	Content	Features	Schools Experimented
1	Mathematical modeling	activity	High School Affiliated to Peking University

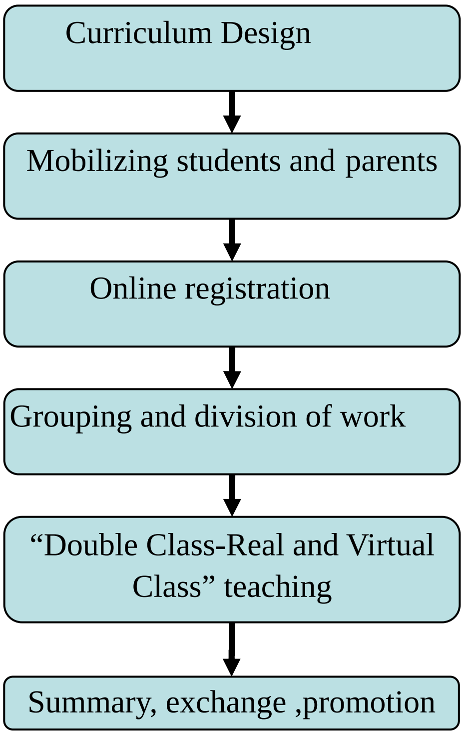
			19th High school in Beijing
			15th High School in Beijing
			Experimental School of Beijing Economic and Technological Development Zone
2	Matrices and transformations	comprehensive	1th High School in Changping
			8th High School in Beijing
			Huiwen High school
3	Solid geometry Introduction	visual	2th high school in Changping
4	Definition and application of derivative	analogy	17th High School in Beijing
			94th High School in Beijing
			Ritan High School in Beijing
5	The optional course of math history	culture	Huiwen High School
			ChenjingLun High School
6	Review and go over the knowledge of function	summary	Qianfeng High School in Changping
7	progression	analogy	Oriental Decai High School
8	conic curve	analogy	1th High School in SanliTun
			ChenjingLun High School
9	probability	other	Foreign Language High School in Chaoyang
10	inequality	other	80th High school in Beijing
11	Reason and argumentation	summary	19th High school in Beijing

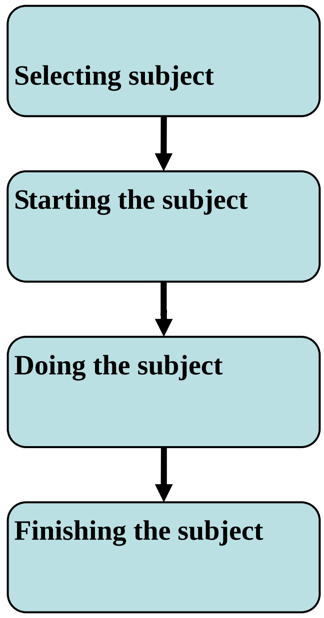
Note: Some school have finished the experiment, some are still carrying it out.

6. Introduction of cases

Due to limited space, we don't show all the cases and details here, but rather only provide two cases for reference. If you would like to check out other cases, please refer to our book.

Case1: mathematical modeling---For "dual class"





Being different from other content, mathematical modeling has a unique teaching procedure--, selecting subject, starting the subject, doing the subject, finishing the subject.

Select subject: During this period, students read the related literature, review the knowledge, discuss in groups, and finally put forward the subject their group want to research.

Start the subject: After several discussions, they have a rough idea and a rough estimation about their job and then submit the report with a preliminary plan and idea on it.

Do the subject: They put the preliminary plan into practice by discussing together, asking for help online, searching for materials, choosing proper tools, writing reports, estimating and calculating. Through their cooperation, a series of achievement will be formed, such as data, formulation, report, software, photos, video, physical model and so on.

Finish the subject: Students show their achievement in the form of papers or reports online and communicate with each other, acquire others' remarks and suggestions to revise and improve their results. In the oral defense conference held in the real class, they can present their work .Teachers guide and give comments which consist of two parts. The first is the quality or accuracy of the mathematical model (the score for the quality).The other is comment which show the distinguish feature and creation, shine point of the work (the score for their creation).

The procedure for dual-class differs due to different content chosen in mathematical modeling. For example, it can be designed like this in function or mathematical modeling:

Learning objectives

1. To deepen understanding about the function by reading articles in the document folder on line.
2. Be able to find several real functions independently
3. Divide into groups, make their own task clear respectively,
4 . Learn how to comment and choose the best mathematical model.

Specific steps and learning requirements

Content and **credit hours**	Learning activities	Learning requirements and recommendations	homework
Real classroom (1 credit hour)	Introduce the mathematical modeling including its characteristics, requirements and function	In order to clear the direction and raise confidence, invite seniors to present their results and feeling.	
Learning Online (1 credit hours)	Students read materials about function	Be able to identify a variety of functions, to deepen understanding of them.	homework 1-1
Finding functions (0.5 credit hours)	Every student can find three real functions independently	Indicate where to find them and what functions they are.	homework 1-2
group sharing online and offline (1 credit hours)	1. Set up learning group. 2. Each group submits the results 3. Exchange and evaluate. 4. Self-evaluation.	1. Elect the leader from the group which includes the members and determine their respective responsibilities. 2. Select the best function this group has done, submit the process and outcome. 3. Select one or two other groups 'results and make their comments. 4. Submit the report, make a summary about the process and results and give the corresponding comment which should include: (1) Whether the whole process is complete, and whether the report for each step has been written. (2) The result is scientific so we can definitely find a function which we are very familiar with. (3) Show their creativity (finding a novel function , or a unique or special	homework 1-3 homework 1-4 homework 1-5

		relationship (4)If the comments on others' work are accurate.	
Identify the subject (0.5 credit hours)	decide the problem they want to solve using a function model	Among the problems provided by teachers ,Select one they want to solve or choose others after several discussion in group	

From the table above, we can see the whole process clearly which can be fixed gradually so that a stable teaching mode for function modeling can be formed.

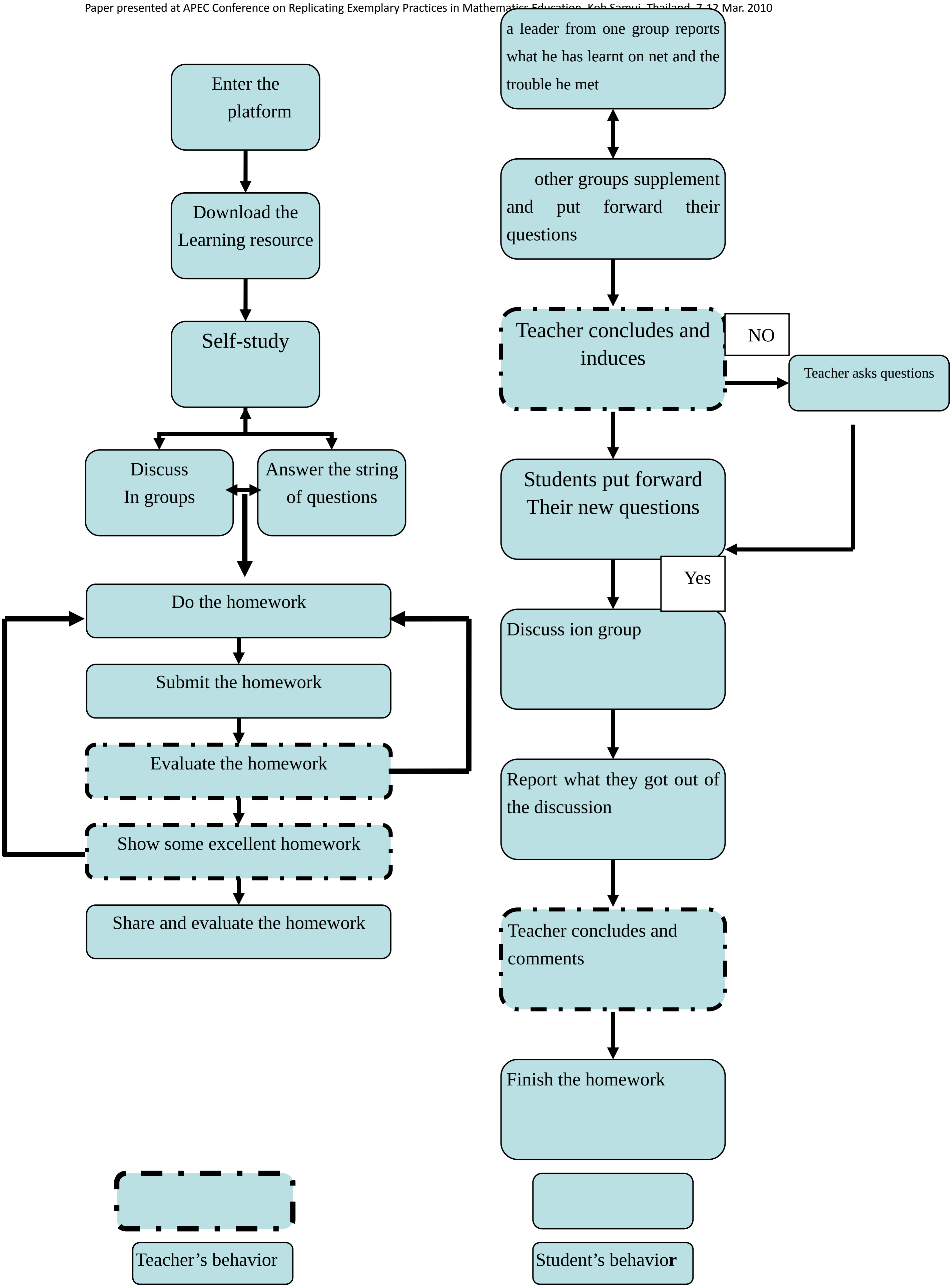
Case 2 "Double Class-Real and Virtual Class" teaching

The compulsory chapter "Solid Geometry Introduction" includes : The features of cylinder, cone, 台体, ball, combination, the drawing method of visual figure, stereogram and deepening the comprehension in the definition ,nature of parallelism or vertical as well as the critical theorem for such relationship between lines and surface in space. We divide them into six sections:

- Learn cuboids and common geometric objects.
- Cuboids and the drawing method of visual figure, stereogram 3.The position or relationship between lines and lines in a cuboid.
- The position or relationship between lines and surfaces in a cuboid.
- The position or relationship between surfaces and surfaces in a cuboid.
- Review cuboids again.

Each section is taught in both real class and virtual class, in which a PPT, a string of tasks, homework and relevant materials are necessary. We design the content in the form question string, which lead students to think by steering them in the right direction, and thus the desired objectives are easily attained.

（1）procedure for virtual class （2）procedure for real class


Enter the platform
Download the Learning resource
Self-study
Discuss In groups
Answer the string of questions
Do the homework
Submit the homework
Evaluate the homework
Show some excellent homework
Share and evaluate the homework
a leader from one group reports what he has learnt on net and the trouble he met
other groups supplement and put forward their questions
Teacher concludes and induces
NO
Teacher asks questions
Students put forward Their new questions
Yes
Discuss ion group
Report what they got out of the discussion
Teacher concludes and comments
Finish the homework
Teacher's behavior
Student's behavior

7. Summary and prospect

Teaching mode for 'dual-class' is still in the exploratory phase and some problems needs to be solved and improved. At present, the main challenges are:

1. What content in the text book can be taught in the dual-class?
2. How to design our new unit curriculum? How to arrange the credit hours? How to design the question string? What kinds of resources are proper?
3. Environment in the virtual class should be improved to make the exchange between students unobstructed.
4. How to input or compile the formula quickly and draw various geometric figures. How to connect mathematical software with network well.

The "Double Class-Real and Virtual Class" has brought great changes for both students and teachers ,break the time and space limit, motivated students' passion and interests in math. Hence, we should believe this new teaching mode and endeavor to spread it. Here are some suggestions:

1. Setting up several schools to be experimented for further explore so that we can make sure what content can be involved in dual-class.
2. Setting up several unit curriculums for dual class to make the new mode easy to spread and resources easy to share.
3. Spreading this new teaching mode gradually to make its value accepted by more teachers and be used in their teaching.

Chinese Reference

『1』 Ministry of Education of the People's Republic of China (MOE).Ordinary High school mathematics curriculum standards (trial version) [M]. People's Education Press, 2003.

『2』 Ye Qixiao. Mathematical modeling teaching activities and Mathematics Education Reform [M]. Hunan Education Press, 2003.

『3』 Shangzhi Wang Unscramble of Mathematics Curriculum Standard of China [M] Jiansu Educational Press 2004

『4』 Shangzhi Wang,etc. Teaching Cases of China [M] Higher Education Press, 2005.

『5』 Zhang Siming Bai Yonhxiao. The practice and exploration of Mathematics learning [M]. Higher Education Press, 2003.

『6』 Development group for mathematics curriculum standards. Interpretation of Full-time compulsory education mathematics curriculum standards (trial version) [M]. Beijing Normal University Press, 2002.

『7』 Development group for mathematics curriculum standards. Interpretation of ordinary high school mathematics curriculum standards (experimental) [M]. Jiangsu Education Press, 2006

『8』 Wang Shangzhi,etc. A series of CD : "Mathematical Modeling Entering into the Classroom in High School” .Central Radio and TV University Audio Visual press 2006.09

『9』 Zhang Siming, etc. New high school mathematical curriculum and learning [M]. Higher Education Press 2008.08

『10』 Wang Shangzhi, etc. Enter into the new high school mathematics curriculum [M]. East China Normal University Press, 2007.09

『11』 Zhang Siming, etc. Teaching Design and practice cases of the project [M]. Beijing Normal University Press 2006.07

『12』 Zhang Siming. Stimulating creativity with mathematics. Mathematical papers 2006 12th edition

『13』 Ma Ping,etc. Let the subject learning become the creative stage. Mathematics journal of secondary school. 2008 1th edition

『14』 John Parker R.L.Moore: Mathematician and Teacher, Column Editor: Osmo

Pekonen Agora Center 40014

『15』 J. R. Kline to R. L. Moore, undated and incomplete letter, Box 20, R. L. Moore Archives, Center for American History, The University of Texas at Austin.

『16』

Zippin, Leo, A study of continuous curves and their relation to the Janiszewski-Mullikin theorem, *Transactions of the American Mathematical Society* **31** (1929), 144-162.

『17』

Zippin, Leo, On continuous curves and the Jordan curve theorem, *American Journal of Mathematics* **52** (1930), 331-350.

『18』

Moore, R. L., Sets of metrical hypotheses for geometry, *Transactions of the American Mathematical Society* **9** (1908), 487-512.

[19] Moore, R. L., A note concerning Veblen's axioms for geometry, *Transactions of the American Mathematical Society* **13** (1912), 74-76.

[20]Moore, R. L., On the foundations of plane analysis situs, *Transactions of the American Mathematical Society* **17** (1916), 131-164.

[20]Moore, R. L., On the foundations of plane analysis situs, *Proceedings of the National Academy of Science* **2** (1916), 270-272.

[21] Moore, R. L., A characterization of Jordan regions by properties having no reference to their boundaries, *Proceedings of the National Academy of Science* **4** (1918), 364-370.

Principles and Processes for Publishing Textbooks and Alignment with Standards: A Case in Japan

Shizumi Shimizu, Teikyo University Tokyo
Tad Watanabe, Kennesaw State University Kennesaw

February 2010

Mathematics textbooks play a central role in mathematics classrooms throughout the world. Shimahara and Sakai (1995) noted that teachers in both the United States and Japan depended on their textbooks to teach mathematics. Although the exact ways teachers use textbooks in teaching mathematics may vary, it is safe to say that textbooks are an important bridge between the intended curriculum, i.e., *standards*, and the implemented curriculum. In this paper, we will discuss how mathematics textbooks are produced in Japan, whose curriculum is often cited as an example of a focused, rigorous, and coherent curriculum.

1 Background

Schools: Japanese schools are divided into three categories: elementary (Grades 1 through 6, 6 years old through 12 years old), lower secondary (Grades 7 through 9, ages 13 through 15), and upper secondary schools (Grades 10 through 12, ages 16 through 18). Elementary and lower secondary schools compose the compulsory education; however, virtually all students go on to upper secondary schools, some of which are specialized high schools focusing on agriculture, commerce, or industrial technologies. Starting in the 2009 Upper Secondary School Course of Study, all 10th graders are required to take Mathematics I instead of selecting either Mathematics I or "Fundamental Mathematics," which was designed for non-college-intending students. Although Mathematics I is to be taken during Grade 10 in general, the new course of study allows some schools to cover the content in 2 or more years.

Teachers: In Japan, in order to teach in elementary and secondary schools, one has to obtain a teaching license issued by the Prefectural Education Board. To obtain the license, a candidate must complete a teacher education program at a college that is approved by the Ministry of Education, Culture, Sports, Science, and Technology (the Ministry of Education, hereafter), and then he or she will have to pass the teaching licensure test offered by the Prefectural Education Board. Of approximately 1 million elementary and secondary school teachers, about 80% are full time faculty members, while the remaining 20% are part-time lecturers. However, a teaching license is required for both groups. More recently, in order to utilize human resources beyond schools, some people without a teaching license have been hired to teach certain courses alongside of licensed teachers.

Japanese elementary school teachers are generalists who teach all subject matters while teachers in lower and upper secondary schools are content area specialists. Thus, at any given elementary school, only about one or two teachers received specialized training in mathematics during their teacher education programs or participate in a mathematics study group is limited. However, since about 90% of elementary school teachers teach mathematics, those teachers with a specialized training play important roles in helping other teachers improve their mathematics teaching.

Textbooks: In Japan, textbooks are published by private publishers based on the *Course of Study* and the accompanying *Teaching Guide*, published by the Ministry of Education. All textbooks must pass through the textbook authorization process overseen by the Textbook Authorization Council. Currently, there are 6 publishers who publish mathematics textbooks for elementary and lower secondary schools -- all 6 publishing textbooks for both levels.

An overview of the textbook production is as follows. First, each publisher develops a draft version, which takes about one year. The draft version is then submitted to the Ministry of Education for its examination. Based on the comments received, the publisher edits the draft version and re-submit the revised version to the Ministry. This process will continue until sufficient revisions are completed. Typically, the textbook authorization process takes about 10 months. Once the textbooks are authorized, the publisher will produce the sample textbooks to be examined by local educational agencies, who will make the final decision on which series is to be adopted. The textbook adoption process takes approximately 6 months, and the new textbooks will be used in schools in the new school year immediately following the adoption decision. Therefore, it takes about 3 years between the time publishers begin producing their textbooks and the time they begin actually being used in classrooms.

As the previous discussion of the textbook production process clearly shows, both commercial publishers and the Ministry of Education play important roles in the production of textbooks in Japan. In the following sections, we will articulate their roles in more detail.

2 Roles of the Ministry of Education

There are three major roles in the production of textbooks in Japan: creation of the Course of Study, publishing the *Teaching Guide*, and the textbook authorization process.

Course of Study: The Course of Study is created by the Ministry of Education. Typically, there is a writing team consisting of about 15 members for each subject area at each school level, and a ministry official oversees the writing process. The writing team members include university professors, school administrators, and classroom teachers. The writing team drafts the course of study for each subject area by carefully examining the recommendations by the Central Education Council. After the Ministry of Education reviews the drafts from all writing teams, examining coherence across subject areas and appropriateness as a legal document, the draft is released to the public for their comments. Upon completion of the public comment period, the final document is released as a law. Typically, this process takes about 2 years.

Because the Course of Study specifies the content at each grade level, it clearly plays a major role in the textbook production. However, it is still a collection of grade-by-grade learning expectation statements. Sometimes the statements in the Course of Study are ambiguous and require further clarification. For that purpose, the Ministry of Education produces a separate document, typically called a *Teaching Guide*.

Teaching Guide: After the Course of Study is publicly released, the Ministry of Education forms committees to draft a *Teaching Guide* for each subject area at each level of schooling. Typically, the members of this committee are the same as the writing team for the course of study. In about a year, the document is published through a private company as it is not a legal document. *Teaching Guide* clarifies the Course of Study by providing additional specifications and examples, as well as rationale for decisions the writing team made. Textbook publishers will examine both the Course of Study and the *Teaching Guide* as they create their textbooks. For both the Course of Study and *Teaching Guide*, go to the APEC HDR Working Group web page, Mathematics Standards in APEC Economies.

Textbook Authorization: The current textbook authorization process has been in place since the end of the World War II. Before then, textbooks were published by the government, and there was only one official textbook series in each subject area. The process of the textbook authorization at the Ministry of Education is as follows. The draft textbooks submitted by commercial publishers, typically called "white-cover edition," they are sent to appropriate Ministry officials and anonymous group of collaborators such as university professors. These reviewers will examine the draft textbooks following the guidelines for textbook authorization established by the Ministry of Education. The Ministry officials organize the comments submitted by the reviewers, then the draft version is officially judged by the Textbook Authorization and Research Council composed of scholars and called by the Minister of Education. The Council will then judge whether or not each series will be authorized, upon completion of required revisions. When publishers make the necessary revisions, the series is officially authorized. In recent years, no textbook series has failed the authorization process.

3 Roles of Commercial Publishers

Organization: The exact process of textbook production varies among publishers. However, most publishers organize a textbook writing committee which will determine the organization and sequencing of the content in each grade and allocation of lessons and the number of pages for each unit. Although the Course of Study dictates the placement of a specific content in a particular grade level, the Course of Study does not specify the sequence of topics within each grade level. Therefore, it is up to each publisher to determine a coherent organization of content within a grade level. The number of lessons in each grade level is determined by the School Law, and publishers typically try to keep the number of lessons in their textbooks to be several lessons fewer so that teachers can plan their instruction more flexibly. The committee will examine the coherence of each grade, within each content domain, and the series as a whole. Actual writing is done by subcommittees, which may be based on grade levels, content domains, or some other considerations.

Authors: Each publisher employs a team of authors to create its mathematics textbook series. These authors may be university professors, mathematics supervisors for local education agencies or classroom teachers. The list of authors is submitted to the Ministry of Education along with the draft version when it is submitted for the Ministry's review. In general, school teachers are prohibited to have a second job, but being a member of textbook authoring team is considered acceptable. Moreover, the fact that textbooks are written by a team that includes a number of experienced classroom teachers, not by professional writers who may or may not have any classroom teaching experiences, ensures that textbooks reflect the reality of classrooms.

Securing good authors is a very important task for any publisher. Therefore, publishers are always looking for young promising teachers, and that is one of the reasons members of editorial department attends various lesson study meetings. Some publishers try to nurture young teachers by engaging them in some preliminary work of textbook writing.

Teachers Manuals: Teachers manuals are purchased by each school. Teachers will use the manuals to construct their daily, unit, and yearly instruction plans. More recently, publishers began to provide some materials on CD or DVD as well as introducing web-based on-line support system. The members of editorial office will attend teacher study groups throughout Japan to monitor how teachers are using their textbooks and listen to their needs, which may be reflected in a newer edition of textbooks and their teachers manuals.

4 Issues and Future Prospects

Textbooks have changed significantly over the past several decades. Those changes reflect the changes in education philosophy and the advance in technologies involved in textbook production. They also reflect the needs and desires of users as well. Moreover, textbooks continue to change. In Japan, several factors seem to be pushing mathematics textbooks to include more and more materials.

In 2003, the Ministry of Education released a partial revision of the Course of Study in response to concerns for declining achievement due to a severe reduction in content and the number of class periods in the 1998 revision. Up till that point, the content of the Course of Study was what every student studied, and textbooks could not include content from later grades. However, with the partial revision of 2003, the Ministry of Education approved the inclusion of advanced materials so that those materials can be used with those students who have mastered the grade-level materials. In other words, not everything in textbooks is for all students. The 2008 Course of Study continues with this practice even though the content and the number of class periods have returned almost to the 1989 level.

Grade	Age (years old)	Standard number of class periods per year in the Course of Study						
		1951	1958	1968	1977	1989	1998	2011
elementary								
1	6	77	102	102	136	136	114	136
2	7	123	140	140	175	175	155	175
3	8	138	175	175	175	175	150	175
4	9	160	210	210	175	175	150	175
5	10	160	210	210	175	175	150	175
6	11	160	210	210	175	175	150	175
Lower secondary								

1	12	140	140	140	105	105	105	140
2	13	140	140	140	140	140	105	105
3	14	140	105-175	140	140	140	105	140

Table 1 Standard numbers of mathematics class periods in the Course of Study from 1951 through 2011.

Furthermore, in recent years, there is an increased interest in textbooks that students can use to study on their own. Traditionally, textbooks were created with the assumption that teachers will use them to guide students' learning. However, more recent textbooks include suggestions to promote students learning more explicitly instead of relying on teachers to provide those suggestions during mathematics lessons. Moreover, with the change in teacher demographics, it is anticipated that the number of less experienced teachers will increase dramatically. Thus, supporting young teachers has become an important consideration for publishers.

In many Japanese classrooms, ancillary materials such as workbooks are used along with textbooks. Some materials are produced by publishers while other materials are produced different publishers, sometimes following a specific publisher's textbook series very closely. The decision to use such materials is made by each school, or sometimes by each individual teacher. However, some believe that textbooks should contain enough materials so that ancillary materials are not needed.

All of the factors discussed above will necessitate an increased amount of content in textbooks and accompanying teachers manuals. Although such an increase may provide teachers with more options, teachers will also have to examine textbooks and make choices carefully.

5 Conclusions and Recommendations

The process of textbook production in Japan, along with some of the issues facing Japanese publishers point to a few suggestions and recommendations to other APEC economies.

Teaching Guide: A curriculum standards is a document reflecting our value judgment. As such, the authors of the document make a number of choices - in what grade should we introduce topic X, how much relative emphasis should be placed on topic Y, etc. The rationale for those choices should be made as explicit as possible so that textbook authors can respect the spirit of the standards. Therefore, we recommend that those who develop mathematics standards to also provide in-depth elaboration of the decisions they made.

For example, in the United States, the Common Core State Standards Initiative is developing a grade-by-grade mathematics standards for Kindergarten through Grade 12. Although the draft standards have not been released at the time of this writing, it is anticipated that the writing team will use its web page to provide more detailed articulation of the standards than what has typically been the case in many U.S. states. It will be useful for publishers, education officials, and classroom teachers to further consider what types of information is particularly useful in supporting mathematics instruction.

Careful Review of Textbooks: Whether this review is conducted by a government agency or by the publishers themselves, it is critical that a textbook series presents mathematics coherently. An extra care should be taken across different levels of schools (elementary, middle/lower secondary, high/upper secondary schools). Some publishers may publish textbooks only at a particular level, thus some forms of an external agency may be needed to conduct such a review. Moreover, those publishers who publish textbooks at only a particular level must carefully examine textbooks at other levels so that students' learning can be smoothly facilitated.

Furthermore, such a review will also consider the coherence with other school subject matters. Mathematics is an important tool for science, and many statistical ideas are used in social studies. For example, the concept of density is an important idea in physical sciences, and it is an example of ratios of two quantities from two distinct measurement fields. Thus, teaching of density in science should be carefully coordinated with the teaching of ratios in mathematics. Students should not encounter a mathematical idea they have yet to learn in other content areas. At the same time, mathematics textbook authors may want to consider how to utilize those situations as the motivation for a new concept.

Classroom Tested: Many curriculum developers conduct a pilot study as they develop their textbook series. However, no textbook is perfect, and they require an on-going revision. Textbook publishers should gather data from those who are using their textbook series to continuously learn things to improve. In the case of Japan, lesson study seems to serve as one mechanism to provide an on-going feedback to improve textbooks. (For more detailed discussion on lesson study, click here.) Research lessons often try to address students' learning difficulties that classroom teachers feel textbooks are not effectively addressing. As noted earlier, members of the textbook editorial office frequently attend these lesson study meetings and observe the lesson and listen to teachers' discussion afterward. Therefore, lesson study provides publishers opportunities to see new ideas in action, not just in theory.

Teacher Learning: Although the quality of textbooks has increased significantly, and although textbooks are no longer limited to the print medium, there are still a variety of constraints that have yet to be overcome. Thus, how well textbooks can support students' learning depends heavily on professional knowledge of teachers who use the textbooks to teach those students. A common saying among Japanese teachers is that "we don't teach the textbook, rather, we teach with textbooks." (See the paper by Takahashi in this monograph for more detailed discussion of this saying.) It is inevitable that new textbook series will include more resources that can support teachers' work. However, that will also mean teachers to make more decisions about the way they will use (or not use) textbooks. Thus, if we want teachers to "teach with textbooks" more effectively, we must support teachers' on-going professional development. Textbook publishers must, therefore, keep teachers' learning in their minds as they produce their textbooks, and particularly the teachers manuals.

Although the appearance of textbooks may change from printed books to multi-media packages, we believe they will still play the central role in mathematics instruction. We hope that this paper will provide useful ideas for those who are responsible for producing textbooks in their own economies.

References

Shimahara, N., & Sakai, A. (1995). *Learning to teach in two cultures*. New York: Garland.

Principles and Processes for Publishing Textbooks and Alignment with Standards: A Case in Singapore

Lianghuo Fan, Nanyang Technological University

March 2010

Abstract: Mainly drawing on the author's experience in textbook development for Singapore schools and research in this area, this paper presents six principles and discusses relevant processes for developing mathematics textbooks. These principles include curriculum principle, discipline principle, pedagogy principle, technology principle, context principle, and presentation principle. For each principle, the author briefly explains what it means, why it is important, and how it can be implemented for the development of mathematics textbooks.

Key Words: Singapore mathematics; Mathematics curriculum; Textbook development

Over the last 15 or so years, Singapore students' outstanding performance in mathematics in large-scale international comparative studies has generated considerable interest among educational researchers and policy makers in its approaches to school mathematics education. Its modern mathematics textbooks, as a most important resource in support of teaching and learning in mathematics classroom, have also received much attention. In fact, Singapore school mathematics textbooks, considered to some extent as exemplary ones, have been adopted, with or without modification, in quite a number of economies over the last decade (e.g., see Quek, 2002). Having said this, I must add that, as researchers have found, there is still much room in Singapore mathematics textbooks for further improvement (e.g., see Ng, 2002; Fan & Zhu, 2007).

In this paper, I shall propose six inter-related principles and discuss relevant processes for developing (publishing) mathematics textbooks. For brevity, these basic principles are termed curriculum principle, discipline principle, pedagogy principle, technology principle, context principle, and presentation principle, respectively, as shown in Figure 1 below. For each principle, I shall briefly explain what it means and/or signifies, why it is important, and how it can be implemented for the development of mathematics textbooks. The discussions are mainly based on relevant research work I, and my co-researchers, have done (e.g., see Fan & Zhu, 2000, 2002, 2007; Ng, 2002; Lee & Fan, 2004), and personal experiences I gained as consultant/editor-in-chief for both primary and secondary mathematics textbooks over the last ten years. In particular, when it is helpful I will draw on examples in the new series of secondary school textbooks, *New Express Mathematics* (see Fan, Cheng, Dong, Leong, Lim-Teo, Ng, et al., 2007, 2008), to illustrate my discussion.

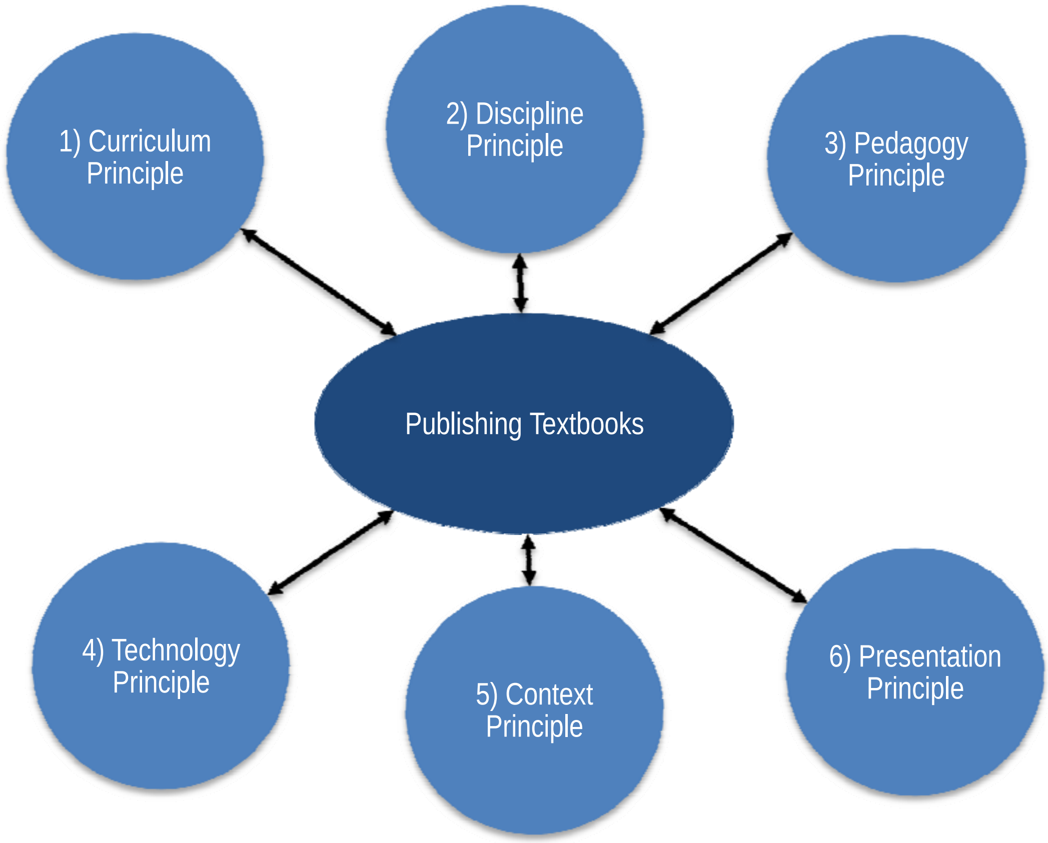


Fig. 1 Six Principles for Developing Mathematics Textbooks

It should be noted that Singapore is an island city-state and has a highly developed economy with a GDP per capita of US$38,904 in 2008 (Source: Singapore Department of Statistics). It has a population of about 5 million and an area of about 700 square kilometers. Chinese, Malay, Tamil, and English are all official languages, but English is the most widely used working language and the medium of instruction in schools. Singapore adopts a highly centralized education system.

Principle 1: Curriculum Principle

The curriculum principle requires that textbooks must be developed for the implementation and realization of intended curriculum.

In a broad sense, curriculum is a course of study, or all the experiences a student will have and achieve in school. School textbook developers (or authors/writers) must have certain "intended" curriculum, explicitly or inexplicitly, in their minds before they develop their textbooks.

In modern societies, because of a variety of reasons and needs, many (if not all) economies have developed so-called national curriculum (syllabus or standards) for school mathematics. In Singapore, like in many other Asian economies, national mathematics curriculum (syllabus) is developed and issued by the Ministry of Education (MOE), and all schools are required to follow the syllabus in teaching, learning, and assessment. Accordingly, textbooks must align themselves with the syllabus. Below is the well-known Singapore mathematics curriculum framework, also known as pentagon framework, stated in the national syllabus (MOE, 2006a, 2006b).

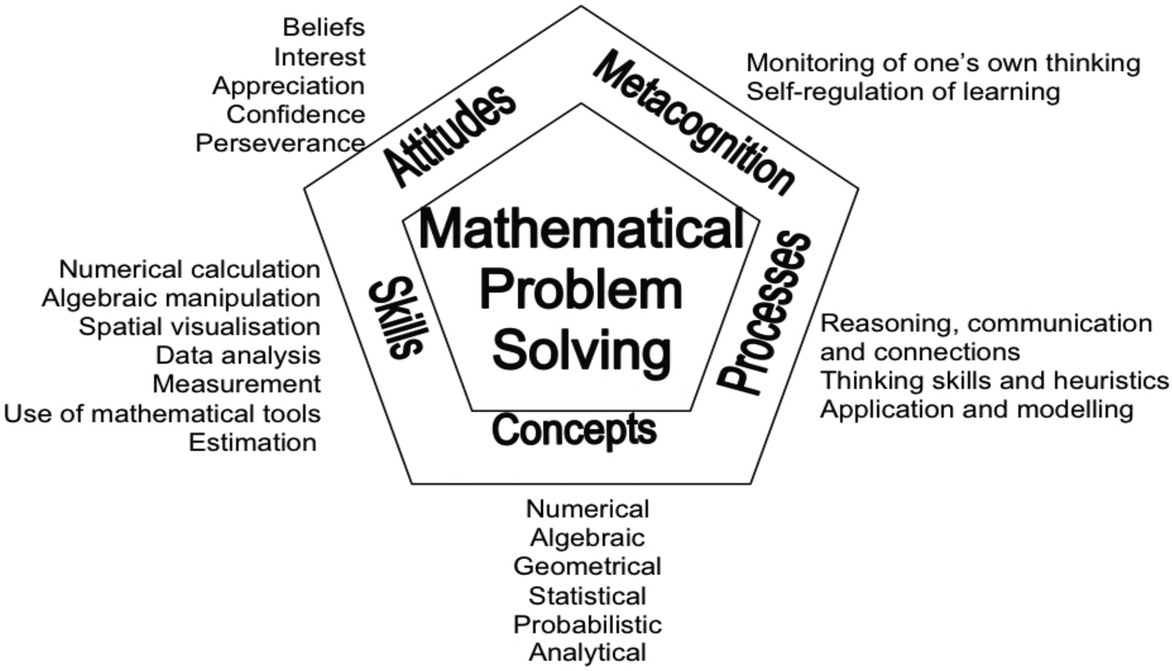


Fig. 2 Singapore Mathematics Curriculum Framework

To align textbooks with the curriculum in terms of the coverage of contents, as roughly reflected in "Concepts" and "Skills" in the pentagon framework and detailed in the syllabus, is important and, relatively, easy. What is more challenging is for textbooks to reflect other aspects that the curriculum intends to achieve, for example, developing students' high-order thinking skills, critical thinking skills and creativity, and positive attitudes towards mathematics, etc. In fact, Ng (2002) found that the whole series of primary textbooks developed by CDIS (see below) only introduced 11 out of the 14 problem solving heuristics listed in the syllabus. Similar inconsistency was also found in secondary mathematics textbooks for the lower grade level, i.e., Grade 7 and Grade 8 (Fan & Zhu, 2007).

As textbooks are essentially textbook developers' own interpretation and reflection of the intended curriculum in the process of textbook development, they must study and hence establish good knowledge of the curriculum, and more importantly, work together and get information and feedback from curriculum developers.

With regard to the alignment of textbooks with national curriculum, as pointed out by Kho, who is a most senior curriculum specialist of MOE, during an interview conducted as a preparation for this paper, an exceptional case in Singapore was that in the 1980s and early 1990s, all the primary mathematics textbooks and a set of secondary mathematics textbooks were developed by two specially appointed teams who also developed the syllabus in the Curriculum Development

Institute of Singapore (CDIS) under MOE (Kho, personal communication, Jan. 31, 2010). In other words, the curriculum developers are also textbook developers[1].

Since the mid 1990s, due to a number of reasons the development and publication of textbooks have been decentralized in Singapore. Nevertheless, the curriculum developers have always maintained very close working connection and interaction with the textbook developers though a variety of activities and channels, including curriculum briefings (e.g., see Mathematics Unit, 2004, 2005), seminars, and meetings. More importantly, all school textbooks in Singapore must be reviewed and approved, primarily based on the curriculum, by a evaluation committee appointed by the Ministry of Education before they can be published and used in schools, and understandably, the curriculum developers have always played a leading role in the evaluation committee. In addition, the textbook developers must revise their textbooks according to the feedback given in the review report. I think these practices have worked very successfully in Singapore and are worth recommendation.

Principle 2: Discipline Principle

There is no doubt that mathematics is a very mature and well-established scientific or academic discipline. The discipline principle requires that school mathematics textbooks must provide solid foundation for the students to understand, apply, and study mathematics in their daily life, further learning and workplace. In terms of content, textbooks must correctly present mathematics knowledge (including mathematical concepts, facts, and methods, etc.). Furthermore, also more challengingly, textbooks should properly represent and reflect the nature, the structure, and epistemology of mathematics as a discipline.

The importance of the discipline principle in developing textbooks is easy to see, but to implement it is not as easy as people might think. Many studies on textbooks have indicated a surprisingly large number of cases in which the textbooks presented the content improperly or incorrectly (e.g., see Levin, 1998).

My own experiences in textbook study and development also suggest that many problems found in textbooks are technical and can be corrected easily, but there are still many which are non-technical or conceptual and they pointed to the problems or weakness in the knowledge base of the textbook developers.

Just to give one case, I shall use an example in the topic of synthetic division in algebra. Many advanced school and college algebra textbooks explicitly stated that this method is only applicable to a divisor in the form of x-a, and it cannot be extended to a divisor being a polynomial with degree higher than 1 (e.g., see Larson & Hostetler, 1997), which is incorrect.

In connection with this topic, to introduce the long division as shown below, some textbooks place the quotient at the top of $0x^3+2x^2-7x-10$ instead of $5x^5+13x^4+0x^3+2x^2$. Although either way will produce the correct answer at this level, the former will hinder students' further learning about how the method can be generalized for other kinds of divisors with degree more

[1] The similar practice also existed in China for a long time, where People's Education Press was the organization both to stipulate the national syllabus and to develop/publish the textbooks based on the syllabus. The situation was changed in the late 1990s, when a new round of curriculum reform was launched.

than 1, and hence latter expression should be used (for more details about the synthetic division and its generalization, see Fan, 2003).

$$
\begin{array}{r l}
 & 5x^3 + 3x^2 + 9x - 7 \\
x^2 + 2x - 3 \,\big)\!\! & \overline{5x^5 + 13x^4 + 0x^3 + 2x^2 - 7x - 10} \\
 & \underline{5x^5 + 10x^4 - 15x^3} \\
 & \qquad 3x^4 + 15x^3 + 2x^2 \\
 & \qquad \underline{3x^4 + 6x^3 - 9x^2} \\
 & \qquad\qquad 9x^3 + 11x^2 - 7x \\
 & \qquad\qquad \underline{9x^3 + 18x^2 - 27x} \\
 & \qquad\qquad\qquad -7x^2 + 20x - 10 \\
 & \qquad\qquad\qquad \underline{-7x^2 - 14x + 21} \\
 & \qquad\qquad\qquad\qquad 34x - 31
\end{array}
$$

From the fact that numerous studies have consistently revealed that many mathematics teachers don't have sufficient knowledge for effective teaching of mathematics (e.g., Carpenter, Fennema, Peterson, & Carey, 1988; Fan, 1998; Ma, 1999), the situation here is not surprising, although it should be noted that virtually no study has been done about what knowledge textbook developers need and have. This is worth attention from researchers as well as policy makers.

With regard to the discipline principle, it is clear that textbook developers must have a sound knowledge base in mathematics as a discipline. It is also very helpful, whenever possible, to have mathematicians in the textbook development team, particularly for secondary and higher level textbooks. In fact, in a latest series of secondary mathematics textbooks (of which I served as chief editor), 10 of the 16 my fellow developers (authors) are trained mathematicians, holding PhD degrees in mathematics from reputable universities. It makes us have more confidence in claiming that one of the key features of the textbooks is, "content is mathematically sound" (Fan, et al., 2007, 2008).

Another relevant point is that textbook developers must carefully collect feedback from the teachers and students after they have used the textbooks. Many times, the problems and mistakes in textbooks cannot be totally detected until they are really used in schools. It implies that textbook development should be ideally an ongoing process.

Having textbooks reviewed, especially by mathematicians and school teachers, is also important in terms of this principle. In Singapore, as Kho pointed out, being mathematically correct is one of the basic criteria for the reviewers to make recommendation for the approval of the textbooks (Kho, personal communication, Jan. 31, 2010), and the textbook developers must correct the mistakes and address the concerns, if any, raised by the reviewers in this aspect.

Principle 3: Pedagogy Principle

The pedagogy principle requires that textbooks must be developed to facilitate the teaching, learning, and assessment in mathematics.

As Fan and Kaeley (2000) indicated, textbooks as a learning tool or resource can convey different pedagogical messages to teachers (and students) and provide them with an encouraging or discouraging curricular environment, promoting different teaching (and learning) strategies. In fact, available studies have consistently revealed, textbooks can, to different extent, affect not only what to teach, but also how to teach, which will ultimately affect students' learning in mathematics (Zhu & Fan, 2002; Fan, Chen, Zhu, Qiu, & Hu, 2004).

In usual, the pedagogical orientation can be provided in the textbooks implicitly, but sometimes it is helpful to make some pedagogical messages explicit. For example, in *New Express Mathematics*, the authors labeled some sections with headings such as "Let us try", "Looking back", "In-class activity", and "Project task" to make the message about the learning and learning process more explicit. For assessment, the textbooks classified mathematics questions into Group A, B, and C. Journal writing tasks and other kinds of so-called alternative assessment tasks are also provided in the textbooks (Fan, et al., 2007, 2008).

Regarding this principle, as found in the case of Singapore, textbook developers are often given more room to be flexible in pedagogical matters. It is important that textbook developers keep abreast with the new development of the practice, theories, and research in pedagogy and learning. It is also very helpful to have mathematics educators and mathematics teachers in the textbook development team, particularly for developing the textbooks for students at lower grade levels. While mathematics educators have strengths in pedagogical theory and research, school teachers often know better the practices and needs of teachers and students in schools. In *New Express Mathematics* mentioned above, all the other authors are mathematics educators, and most of them have school teaching experiences (Fan, et al., 2007, 2008).

It is worth mentioning that in the process of developing the primary mathematics textbooks by CDIS in the 1980s, as said earlier, all the content and activities designed by the project team, which was led by Kho, were piloted in classrooms in a number of schools which volunteered to participate in the trial, and then revised according to the feedbacks from the try-out before they were finalized and published. According to Kho (personal communication, Jan. 31, 2010), this process was unique and very effective for the developers to make sure that the textbooks being developed would be suited to the needs of teaching and learning in classrooms.

Having textbook reviewed by mathematics educators, or pedagogical experts, and school teachers before the textbooks are published and listening to teachers' feedback after publication is also important for improvement with respect to the pedagogy principle.

Principle 4: Technology Principle

The meaning of technology in mathematics education has expanded over the time, from calculator, to calculator and computer, and now more commonly to information and communication technology (ICT).

About 15 years ago, I criticized, with good intention, that mathematics education including textbooks in China was largely isolated from modern technology and there was virtually no existence of technology in the mathematics textbooks (Fan, 1995). I must say that this criticism is no longer valid, as China has made dramatic progress in this aspect in the new wave of curriculum reform, most visibly in the new textbooks developed. In Singapore, much progress has also been made over the last decade or so.

Undoubtedly, the advent of modern technology has produced significant influences on our modern society. In the field of mathematics education, technology has affected what to teach and how to teach, and moreover, why to teach. In relation to this, technology must be reflected and, more importantly, embedded into the teaching and learning of mathematics. Textbooks, as a most important pedagogical resource, must integrate technology to support and facilitate the teaching and learning of mathematics. With the rapid development of technology, it appears apparent that technology will play an increasingly important role in the next generation of mathematics textbooks.

Let me briefly share some examples in the case of *New Express Mathematics* to illustrate how the technology principle is, to different extents, reflected in the textbook development. The first example is, in the older mathematics textbooks of which I was also a consultant/general editor, the approximate value of π, most commonly 22/7 and sometimes 3.142, was provided mainly for easy calculation. This is no longer the case in the new textbooks, because all students are expected to use calculators, in which keying in 22/7 is not only redundant and less accurate, but also less efficient than directly keying in the symbol “π” or “pi” (similar idea applies to other special values in mathematics, e.g., e).

Another example is that, as all students in Singapore are expected to have access to ICT including calculators, computer and internet, the textbooks developers have developed more authentic and challenging problems including investigative and project tasks. In working on these problems, students will focus more on conceptual understanding, information gathering, logical reasoning and data analysis, and so on, rather than tedious calculation, complex algebraic manipulation, or time-consuming drawing, etc. By doing so, technology can make mathematics teaching and learning not only more efficient, but also more effective. In fact, many questions in the textbooks that are targeted to develop students’ high-order thinking and problem solving abilities are ICT-embedded.

In addition, many topics covered in the textbooks, especially those in geometry (e.g., for measuring and construction) and statistics (e.g., for statistical diagrams, graphical representation and data analysis), were introduced with the use of available mathematics software to facilitate students’ learning (Fan, et al., 2007, 2008).

The technology principle requires that the textbook developers be familiar with the development of technology. In particular, having experts in the use of ICT in mathematics teaching and learning on board would be most helpful in this aspect. In addition, feedback from teachers and students is also helpful with regard to this principle.

Finally, I shall very briefly mention context principle and presentation principle. Although I think in some sense they are less important and more technical compared to these described earlier, they are still worth reasonable attention in textbook development.

Principle 5: Context Principle

School mathematics textbooks are not research publications for pure mathematics, which can be almost completely abstract. School mathematics is often contextualized, and cannot be free from the social and cultural background, under which school education takes place.

The context principle requires the textbook developers provide adequate cultural, social and even historical contexts when introducing mathematics concepts and contents. This principle is particularly important when application of mathematics is concerned. In the case of *New Express Mathematics*, many examples and problems use Singapore's local context as background. For example, they use authentic information about Singapore's geography (e.g., for distance, speed, and time), demography and economy (e.g., for statistics), architecture (e.g., for geometrical shapes), and society (e.g., social welfare and public housing system for financial mathematics). The contextualized information is provided to motivate and engage students in their learning of mathematics as they are familiar and can make connection with the contexts. In this sense, as it is found in the case of Singapore mathematics textbooks, modification or localization of the textbooks is necessary when they are developed in one economy but used in other economies.

The context principle requires the textbook developers have reasonable knowledge of local contexts. Having local mathematics education experts and school teachers in the development team is important in this aspect. Searching information from local newspapers and other sources can also be very helpful.

Principle 6: Presentation Principle

This principle requires that the presentation of the contents in textbooks must suit the level and needs of teaching and learning. This principle is meaningful in the textbook development as well-designed presentation can make the reading and use of textbooks easy and pleasant, and facilitate teaching and learning.

The principle is more about the technical aspects of developing and publishing a textbook, "design and physical features". In Singapore, the mathematics unit of the Ministry of Education once recommended the following four aspects for textbook developers/publishers to consider: 1. Real-life pictures and realistic drawings, 2. Clear layout and illustrations, 3. Use of colors, and 4. Simple language. (Mathematics Unit, 2004).

Largely consistently, in developing the series of *New Express Mathematics*, the developers' guidelines in this aspect were "1. Use clear and concise language to describe mathematics concepts and process, so it is easy for students to understand; and 2. Use diagrams, pictures, and other visual representations, whenever possible, to make the textbooks more interesting and visually appealing to students and hence enrich and enhance students learning experiences in mathematics" (New Express Mathematics Project Team, 2004).

To implement this principle, textbook developers and publishers should work together (and share the strengths and responsibilities), have experts or specialists in relevant areas, and most importantly, pay reasonable attention to the aspects as highlighted above.

The following table provides a summary of the principles and process/recommendations for developing mathematics textbooks, as presented and discussed in the article. Readers are reminded again that they are based on my own experience and mainly with a Singapore context.

Table 1

A Summary of Principles and Processes for Publishing Mathematics Textbooks

Basic Principles	Processes/Recommendations
Curriculum Principle: Textbooks must be developed for the implementation and realization of intended curriculum.	1. Textbook developers have good knowledge of the curriculum. 2. Textbook developers closely work together and have interaction with curriculum developers. 3. Textbooks be reviewed by reviewers including curriculum developers.
Discipline Principle: Textbooks must provide solid foundation for the students to understand, apply, and study mathematics.	1. Textbook developers have a sound knowledge base in mathematics. 2. Development team include mathematicians. 3. Textbook developers collect ongoing feedback from the users. 4. Textbooks be reviewed by reviewers including mathematicians and school teachers.
Pedagogy Principle: Textbooks must be developed to facilitate the teaching, learning, and assessment in mathematics.	1. Textbook developers have good knowledge in pedagogy. 2. Developer team include mathematics educators and school teachers. 3. Textbook developers collect ongoing feedback from the users. 4. Textbooks be reviewed by reviewers including pedagogical experts and school teachers.
Technology Principle: Textbooks must integrate technology to support and facilitate the teaching and learning of mathematics.	1. Textbook developers be familiar with the development of technology. 2. Developer team include experts in the use of ICT in mathematics teaching and learning. 3. Textbook developers collect feedback from the users.
Context Principle: Textbooks must provide adequate cultural, social and even historical contexts when introducing mathematics concepts and contents.	1. Textbook developers have good knowledge of local contexts. 2. Development team include local mathematics experts and school teachers.
Presentation Principle: Textbooks must suit the level and needs of teaching and learning.	1. Textbook developers and publishers work together. 2. Development team include experts in relevant areas.

References

Carpenter, T. P., Fennema, E., Peterson, P. L., & Carey, D. A. (1988). Teachers' pedagogical content knowledge of students' problem solving in elementary arithmetic. *Journal for Research in Mathematics Education*, *19*, 385-401.

Fan, L. (1995). A review of the recent development of mathematics education in the U.S. and its comparison with China. *Mathematics Teaching* (Shanghai, China), 140, 1-3 & 141, 4-6.

Fan, L. (1998). *The development of teachers' pedagogical knowledge: An investigation of mathematics teachers in three high-performing high schools*. Ann Arbor, MI: UMI Dissertation Service, No. 9841511.

Fan, L., Cheng, W., K., Dong, F. M., Leong, Y. H., Lim-Teo, S. K., Ng, W. L., Quek, K. S., Tang, W. K., Tay, E. G., Teo, K. M., Toh, T. L., Yap, S. F., Yeap, B. H., Zhao, D. (2007). *New Express Mathematics* (Book 1 & 3). Singapore: Multimedia Communications.

Fan, L., Cheng, W., K., Dong, F. M., Leong, Y. H., Lim-Teo, S. K., Ng, W. L., Quek, K. S., Tang, W. K., Tay, E. G., Teo, K. M., Toh, T. L., Yap, S. F., Yeap, B. H., Zhao, D. (2008). *New Express Mathematics* (Book 2 & 4). Singapore: Multimedia Communications.

Fan, L., & Kaeley, G. (2000). The influence of textbooks on teaching strategies: An empirical study. *Mid-Western Educational Researcher*. *13*(4), 2-9.

Fan, L., & Zhu, Y. (2000). Problem solving in Singaporean secondary mathematics textbooks. *The Mathematics Educator*. *5*(1/2), 117-141.

Fan, L. (2003). A generalization of synthetic division and a general theorem of division of polynomials. *Mathematical Medley*. *30*(1), 30-37.

Fan, L., Chen, J., Zhu, Y., Qiu, X., & Hu, Q. (2004). Textbook use within and beyond Chinese mathematics classrooms: A study of 12 secondary schools in Kunming and Fuzhou of China. In L. Fan, N. Y. Wong, J. Cai, & S. Li (Eds.). *How Chinese learn mathematics: Perspectives from insiders* (pp. 228-261). Singapore: World Scientific.

Fan, L., & Zhu, Y. (2007). Representation of problem-solving procedures: A comparative look at China, Singapore, and US mathematics textbooks. *Educational Studies in Mathematics*. *66*(1), 61-75.

Larson, R. E., & Hostetler, P. R. (1997). *Algebra and Trigonometry* (4th ed.), Boston: Houghton Mifflin.

Lee, P. Y., & Fan, L. (2004). The development of Singapore mathematics curriculum: Understanding the changes in syllabus, textbook and approaches. In P. Y. Lee, D. Zhang, & N. Song (Eds.), *Proceedings of the International Congress of Mathematicians (ICM 2002) Satellite Conference on the Reform of Mathematics Curriculum and its Education in the 21st Century* (pp. 31-36). Chongqing, China: Chongqing Publishing House.

Levin, S. W. (1998). Fractions and division: Research conceptualizations, textbook presentations, and student performances (doctoral dissertation, University of Chicago, 1998). *Dissertation Abstracts International 59*, 1089A.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.

Mathematics Unit. (2004). 1st briefing to publishers on O-level mathematics. Science Branch, Curriculum Planning and Development Division, Ministry of Education, Singapore.

Mathematics Unit. (2005). 2nd briefing to publishers on O-level mathematics. Science Branch, Curriculum Planning and Development Division, Ministry of Education, Singapore.

Ministry of Education. (2006a). *Mathematics syllabus (Primary)*. Singapore: Author.

Ministry of Education. (2006b). *Mathematics syllabus (Secondary)*. Singapore: Author.

New Express Mathematics Project Team. (2005). Conceptual Framework for New Express Mathematics. Internal Working Document (Available upon request)

Ng, L. E. (2002). *Representation of problem solving in Singaporean primary mathematics textbooks with respect to types, Pólya’s model, and heuristics*. Unpublished master’s thesis, National Institute of Education, Singapore.

Quek, T. (2002). US, Malaysia, Thailand, Viet Nam, India, Pakistan, Bangladesh, Finland … Now, Israel uses S'pore maths textbooks too. *The Straits Times*, Sept. 23, 2002, Singapore.

Zhu, Y., & Fan, L. (2002). Textbook use by Singaporean mathematics teachers at lower secondary level. In D. Edge & B. H. Yap (eds.), *Mathematics education for a knowledge-based era* (Vol. 2) (pp. 194-201). Singapore: Association of Mathematics Educators.

Prospective and Practicing Teacher Professional Development with Standards

Akihiko Takahashi, Ph.D., Depaul University

February 2010

Introduction

It is obvious that teachers cannot teach mathematics beyond their knowledge (National Mathematics Advisory Panel, 2008) but even having this knowledge is not nearly enough to teach mathematics effectively. In order to promote high quality mathematics education for all, ministers of the Asia-Pacific Economic Cooperation (APEC) released a strategic action plan and recommendations for priorities of mathematics and science education[1]. The recommendations recognize the needs of teachers with strong knowledge and expertise in providing high-quality learning opportunities for their students.

In order to promote high quality mathematics education for all, it is critical for universities and school systems to provide both prospective and practicing teachers with opportunities not only to increase their knowledge for teaching mathematics, but also to develop the expertise for teaching mathematics.

In this paper, I discuss the roles of universities and school systems in providing high-quality learning experiences for prospective and practicing teachers -- establishing a strong foundation for teaching mathematics for future generations.

Issues in teaching mathematics

One of the major challenges in mathematics education is the reliable implementation of insights gained from research into the classroom. Despite the fact that researchers have developed great ideas and resources for teaching mathematics, Stigler & Hiebert (2009) argue that the substantive nature of what happens in classrooms has not been changed much.

One of the reasons for this phenomenon may be the lack of the opportunities for prospective and practicing teachers to develop expertise in using ideas from research in their teaching practice. As Polya begins his famous book, *How to Solve It* (1945), helping students to learn mathematics demands time, practice, devotion, and sound principles. Unfortunately many school systems do not have adequate supporting structures for their teachers to develop knowledge and expertise for supporting their students in learning mathematics. As a result, many educators are essentially teaching the same way they were taught in school (Conference Board of the Mathematical Sciences Washington DC. National Advisory Committee on Mathematical Education.[BBB12494], 1975).

In order to bring ideas from research into the classroom, thereby improving teaching and learning mathematics, providing teacher preparation programs for prospective teachers is not enough. Continuous professional development for practicing teachers is also important. Therefore, universities and school systems should be the place for supporting both prospective

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[1] http://hrd.apecwiki.org/index.php/4th_APEC_Education_Ministerial_Meeting_%28AEMM%29_in_Lima_Peru

and practicing teachers in developing knowledge and expertise for their students to learn mathematics.

Resources to support developing knowledge and expertise required for teaching mathematics

Once standards are developed, researchers, curriculum coordinators, and textbook authors and publishers carefully align the curriculum and design materials for implementing the standards. Although it is essential to have a set of good curriculum materials, including textbooks, manipulatives, technological tools and workbooks, developing resources to help teachers develop deeper understandings of the standards and the curriculum materials is also important. When developing such resources for prospective and practicing teachers, it is critical to recognize what knowledge and expertise is necessary for teachers to implement the standards into every day classrooms.

An idea shared among Japanese mathematics educators gives us a framework to examine in terms of teachers' knowledge and expertise for teaching mathematics.

Although most teachers use textbooks as their primary instructional materials (Shimahara & Sakai, 1995; Sugiyama, 2008), Japanese teachers and educators recognize that there are different ways to use textbooks and these ways are significant for student learning. The educators emphasize a distinction between "teaching the textbook" and "teaching mathematics using the textbook." To teach the textbook, teachers need little knowledge about mathematics; they can simply tell students what is in the textbook. However, to teach mathematics *using* the textbook, teachers need to possess a much deeper understanding of mathematics and how students learn mathematics.

In order to provide better learning experiences for students, all teachers should be able to teach mathematics using the textbook effectively. "Teaching the textbook" is not enough. What knowledge and expertise are Japanese teachers expected to develop in order to use the textbook effectively? When and how do Japanese prospective teachers and novice teachers acquire that knowledge and expertise?

Three levels of teaching

Knowing the content in textbooks is the most important foundation in order to be a teacher, however it is not enough to be an effective teacher. Japanese mathematics educators and teachers understand that there exist several levels of teaching between "teaching the textbook" and "teaching mathematics by using the textbook". Japanese mathematics educators typically characterize teacher expertise according to three levels (Sugiyama 2008):

- Level 1: Teachers can tell students important basic ideas of mathematics such as facts, concepts, and procedures.
- Level 2: Teachers can explain the meanings of and reasons behind the important basic ideas of mathematics in order for students to understand them.
- Level 3: Teachers can provide students opportunities to understand these basic ideas, and support their learning so that the students become independent learners.

Although it is essential for teachers to be able to tell students important facts, a teacher at Level 1 is not yet considered a professional. Sugiyama (2008) writes that during the early 20th century, which is considered an early stage of the Japanese public education system, most elementary school teachers were at Level 1. They told their students the facts and expected them to memorize those facts through practice. Textbooks at that time were designed to support this form of instruction.

Level 2 teachers have to know mathematics beyond what is used in everyday life or what is required to solve problems in elementary school textbooks. For example, it is enough for a Level 1 teacher to know that, when dividing fractions, a quotient can be found by multiplying the reciprocal of a fraction. However, Level 2 teachers should be able to explain how multiplying by the reciprocal of a fraction produces the quotient. This type of knowledge is important for helping students understand mathematics. Japanese mathematics educators consider that a teacher at Level 2 can be a considered a professional.

Although Level 2 teachers are considered professionals, Japanese mathematics educators believe that all teachers of mathematics should be at Level 3. This is because Level 2 teachers cannot provide adequate opportunities for students to develop proficiency with understanding.

The differences between Level 3 teachers and other levels can be understood by looking at how they might use a problem in a textbook. A Level 1 teacher would present the problem and show the steps for solving it. A Level 2 teacher would show the steps and explain why those steps are correct and useful. A Level 3 teacher, in contrast, would present students with the same problem, providing structure and guiding the conversation, so that students arrive at a new understanding as a result of their own efforts in solving it. The philosophy behind Level 3 teaching is that students should have reasonable independent work, such as problem solving, in order to develop knowledge, understanding, and skill of mathematics (Lewis & Tsuchida, 1998; J. Stigler & Hiebert, 1999; Akihiko Takahashi & Yoshida, 2004; Yoshida, 1999).

Therein lies the distinction between “teaching the textbook” and “teaching mathematics using the textbook.” Since Level 3 teaching clearly requires greater knowledge and expertise beyond knowing and being able to use mathematics in practical situations, the following question still remains: What professional development programs do teachers need to develop such knowledge and expertise?

Two major types of professional development

When designing professional development programs for prospective and practicing teachers, it is useful to recognize that the professional development programs might be categorized into two types: Phase 1 and Phase 2.

Phase 1 professional development focuses on developing knowledge for teaching mathematics: content knowledge of mathematics, pedagogical content knowledge for teaching mathematics, curricular knowledge for designing lessons, and general pedagogical knowledge (Fernandez, Chokshi, Cannon, & Yoshida, 2001; Lewis, 2000; Lewis & Tsuchida, 1998; J. Stigler & Hiebert, 1999; A. Takahashi, 2000; Yoshida, 1999). In order for teachers to develop such knowledge, this type of professional development usually provides teachers opportunities to learn through reading books and resources, listening to lectures, and watching visual resources such and video and demonstration lessons.

Phase 2 professional development, on the other hand, focuses on developing *expertise* for teaching mathematics: skill for developing lessons for particular students, questioning techniques, skill for designing and implementing formative assessments, foresight for anticipating students responses to questions, and skill for purposeful observation of students during a lesson. To develop such expertise for teaching, teachers should plan a lesson carefully, teach the lesson based on the lesson plan, and reflect upon the teaching and learning based on careful observation. Japanese teachers and educators usually go through this process using *Lesson Study* (Firestone, 1996; Huberman & Guskey, 1994; Little, 1993; Miller & Lord, 1994; Pennel & Firestone, 1996).

Japanese lesson study model

The practice of lesson study originated in Japan. Widely viewed as the foremost professional development program, lesson study is credited with dramatic success in improving classroom practices for the Japanese elementary school system (Lewis, 2002b).

Lesson study embodies many features that researchers have noted are effective in changing teacher practice, such as using concrete practical materials to focus on meaningful problems, taking explicit account of the contexts of teaching and the experiences of teachers, and providing on-site teacher support within a collegial network. It also avoids many features noted as shortcomings of typical professional development, e.g., that it is short-term, fragmented, and externally administered (Akihiko Takahashi & Yoshida, 2004).

Lesson study promotes and maintains collaborative work among teachers while giving them systematic intervention and support. During lesson study, teachers collaborate to: 1) formulate long-term goals for student learning and development; 2) plan and conduct lessons based on research and observation in order to apply these long-terms goals to actual classroom practices for particular academic contents; 3) carefully observe the level of students' learning, their engagement, and their behaviors during the lesson; and 4) hold debriefing sessions with their collaborative groups to discuss and revise the lesson accordingly (Shulman, 1986).

One of the key components in these collaborative efforts is "the research lesson," in which, typically, a group of instructors prepares a single lesson, which is then observed in the classroom by the lesson study group and other practitioners, and afterwards is analyzed during the group's debriefing session. Through the research lesson, teachers become more observant and attentive to the process by which lessons unfold in their class, and they gather data from the actual teaching based on the lesson plan that the lesson study group has prepared. The research lesson is followed by the debriefing session, in which teachers review the data together in order to: 1) make sense of educational ideas within their practice; 2) challenge their individual and shared perspectives about teaching and learning; 3) learn to see their practice from the student's perspective; and 4) enjoy collaborative support among colleagues.

A framework for designing programs for prospective and practicing teachers

Providing a variety of effective programs and usable resources for prospective and practicing teachers is an important role for universities and school systems. At the same time, it is also important to consider how and when these resources should be provided to the prospective and practicing teachers. Some resources may be appropriate for prospective teachers to help them develop a substantial pedagogical knowledge for understanding a standards-based curriculum.

Some resources might be more useful for developing expertise after the teachers have acquired basic pedagogical skills. Providing all the resources during a prospective teacher program might not be the most efficient way for teachers to use these resources effectively. Some of the resources might be more effective after the teachers gain several more years of experience following their teaching experience in a lesson study.

In order to do so, the first step in designing the programs and resources is to develop a framework to identify the purpose and the target audiences of each program and resource.

Based on the earlier discussion contrasting teacher knowledge and expertise, the three levels of teaching, and the two types of professional development, I propose the following matrix to provide a framework for developing programs and resources for mathematics teacher education:

Table 1: A framework for developing programs and resources for mathematics teacher education

<table>
<tr><th></th><th>To become a Level 1 teacher</th><th>To become a Level 2 teacher</th><th>To become a Level 3 teacher</th></tr>
<tr><td>Phase 1
Professional
Development</td><td>Strengthen knowledge of mathematics…

…through:
• Studying textbooks and workbooks
• Using online resources and courses</td><td>Acquire knowledge of mathematics teaching and learning—
• Pedagogical content knowledge
• Knowledge of the curriculum
• Knowledge of the students
• Knowledge of pedagogy…

…through:
• University courses
• Professional development workshops
• Online resources
• Classroom videos
• Classroom observations, including participating in research lessons</td><td>Update knowledge of mathematics teaching and learning…

…through:
• Workshops
• Evening and summer coursework</td></tr>
<tr><td>Phase 2
Professional
Development</td><td></td><td>Understand the process of lesson study …

…through:
• Designing mock-up research lessons as part of university coursework
• Lesson study during student teaching</td><td>Develop expertise for teaching …

…through Lesson Study</td></tr>
</table>

<u>Phase 1 for Level 1</u>

Level 1 is the foundation for becoming a teacher of mathematics, since one cannot teach mathematics if one does not know the content. Usually prospective teachers who come to a university or a teacher-training institute already possess the basic knowledge required for Level 1 teaching. If this is not the case, there should be programs to review content knowledge, such as through online courses or individual tutoring. Although they might be needed for only a small number of prospective teachers, such programs could help more people become teachers. Online courses and resources might be appropriate since the target audience may be smaller number but geographically widely spread.

Phase 1 and Phase 2 for Level 2

Developing knowledge and expertise for Level 2 teaching should be the major focus of university teacher training programs for prospective teachers. Since knowing the content of mathematics is not enough, Level 2 teaching requires the knowledge beyond being able to solve mathematics problems for elementary and middle school students. For example, to teach the formula for finding the area of a parallelogram, Level 2 teachers must know how the formula was developed, why the formula works for any parallelogram regardless its size and orientation, and how the formula is related to other formulas for finding the area of basic geometric shapes.

The knowledge required for Level 2 teaching is a special kind of knowledge for mathematics teachers, and is often called pedagogical content knowledge (Shulman, 1986). Since the knowledge is only required for teaching mathematics, universities and teacher-training institutes should design special courses and resources for prospective teachers of mathematics. In other words, providing regular university level mathematics courses is not sufficient and not appropriate for prospective teachers. Providing dedicated courses and resources for prospective teachers should be the major focus of Phase 1 professional development in preparing Level 2 teachers. At the same time, prospective teachers should develop an understanding of what a good lesson looks like and how to design lessons.

Phase 2 professional development in Level 2 teaching should focus on introducing the idea and the process of lesson study. Engaging in lesson study offers teacher candidates not only practice in developing lessons and teaching lessons based on a plan, but also practice in observing students' learning processes and reflecting upon a lesson.

Phase 1 and Phase 2 for Level 3

Achieving Level 3 is quite demanding and requires extensive Phase 2 professional development. It is essential to understand the philosophy of teaching and learning mathematics, to develop a vivid image of the ideal mathematics class as a model, and to know key instructional techniques for enabling students to learn mathematics independently. Most knowledge and understanding for Level 3 teaching may be obtained through Phase 1 professional development programs such as reading books, listening to lectures, and observing well-designed mathematics classes. However, acquiring the knowledge and understanding is not sufficient to develop the expertise needed for Level 3 teaching. To develop this expertise requires considerable teaching experience, with reflection. Japanese teachers and researchers work collaboratively through lesson study to develop expertise for Level 3 teaching.

Recommendation for universities and school systems

Recognize that knowing mathematics is not enough to help students learn mathematics

Some people still believe that anyone can be a teacher if he or she knows enough mathematics, and therefore teachers do not need any special training to be and to continue being teachers. One of the first steps toward having effective mathematics teachers in the classroom is to help policy makers and leaders in society who have the opportunity recognize the needs for establishing supporting structures not only for prospective teachers but also for practicing teachers.

Research

Research is essential to the design of programs and resources provided for teachers. The first step toward establishing effective programs and usable resources would be to study the needs of the prospective and the practicing teachers. This could be accomplished by using the proposed framework for developing programs and resources for mathematics teacher education. Once the programs and resources are established, the next step would be to examine their effectiveness through empirical research. Since the ultimate goal of these programs and use of resources is to promote better mathematical skills and understandings for the students, the research project would require substantial time and effort. Although research might not be able to contribute to immediate results of the university's efforts, actionable research should always be the foundation of the decision making for world-class universities.

Resources and programs

After establishing effective programs and useful resources, universities and school systems traditionally provide these only to their enrolled students and teachers. The concept of open courseware[2] is to share high quality educational materials with a wider audience. A collaboration of more than 200 higher education institutions and associated organizations from around the world established the Open Courseware Consortium and created a broad open educational content using a shared model. In fact, the APEC Human Resource Development Working Group uses the concept of the open courseware for the Knowledgebank web site using Wiki technology[3].

2 http://www.ocwconsortium.org/

3 http://hrd.apecwiki.org/index.php/Main_Page

References

Conference Board of the Mathematical Sciences Washington DC. National Advisory Committee on Mathematical Education.[BBB12494]. (1975). *Overview and Analysis of School Mathematics, Grades K-12*.

Fernandez, C., Chokshi, S., Cannon, J., & Yoshida, M. (2001). Learning about lesson study in the United States. In E. Beauchamp (Ed.), *New and old voices on Japanese education*. New York: M. E. Sharpe.

Firestone, W. A. (1996). Images of teaching and proposals for reform: A comparison of ideas from cognitive and organizational research. *Educational Administration Quarterly, 32*(2), 209-235.

Huberman, M., & Guskey, T. T. (1994). The diversities of professional development. In T. R. Guskey & M. Huberman (Eds.), *Professional development in education: New paradigms and practices*. New York: Teachers College Press.

Lewis, C. (2000, April 2000). *Lesson Study: The core of Japanese professional development.* Paper presented at the AERA annual meeting.

Lewis, C. (2002b). *Lesson study: A handbook of teacher-led instructional improvement*. Philadelphia: Research for Better Schools.

Lewis, C., & Tsuchida, I. (1998). A lesson like a swiftly flowing river: Research lessons and the improvement of Japanese education. *American Educator*, *22*(4).

Little, J. W. (1993). Teachers' professional development in a climate of educational reform. *Educational Evaluation and Policy Analysis, 15*(2), 129-151.

Miller, B., & Lord, B. (1994). *Staff development for teachers: A study of configurations and costs in four districts*. Newton, MA: Education Development Center.

National Mathematics Advisory Panel. (2008). *Foundations for Success: The Final Report of the National Mathematics Advisory Panel*. Washington, DC: U.S. Department of Education.

Pennel, J. R., & Firestone, W. A. (1996). Changing classroom practices through teacher networks: Matching program features with teacher characteristics and circumstances. *Teachers College Record*, *98*(1).

Polya, G. (1945). *How to solve it: A new aspect of mathematical method*. Princeton, New Jersey: Princeton University Press.

Shimahara, N., & Sakai, A. (1995). *Learning to teach in two cultures*. New York: Garland.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Stigler, J., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: Free Press.

Stigler, J. W., & Hiebert, J. (2009). Closing the Teaching Gap. *Phi Delta Kappan, 91*(03), 32-37.

Sugiyama, Y. (2008). *Introduction to elementary mathematics education*. Tokyo: Toyokan publishing Co.

Takahashi, A. (2000). Current trends and issues in lesson study in Japan and the United States. *Journal of Japan Society of Mathematical Education, 82*(12), 15-21.

Takahashi, A., & Yoshida, M. (2004). How Can We Start Lesson Study?: Ideas for establishing lesson study communities. *Teaching Children Mathematics, Volume 10, Number 9.*, pp.436-443.

Yoshida, M. (1999). *Lesson study: A case study of a Japanese approach to improving instruction through school-based teacher development.* Unpublished Dissertation, University of Chicago, Chicago.

Rating Teachers and Rewarding Teacher Performance: The Context of Singapore

Christine Kim-Eng Lee,[1] Nanyang Technological University
Mei Ying Tan,[2] Columbia University

February 2010

Introduction

Singapore's education system has received international recognition as one of the top performing systems in the world. The McKinsey study of 10 top performing school systems found three attributing factors that are important: "getting the right people to become teachers; developing them into effective teachers; and ensuring that the system is able to deliver the best possible instruction for every child". Teacher quality made the largest difference in student achievement and the most effective school systems invested in their teachers (McKinsey, 2007).

Singapore continues to invest heavily in education and there was an increase of 5.5% in the 2009 budget over 2008 providing a total of S$8,701 million which is 3.49% of its Gross Domestic Product in spite of the economic downturn (Ministry of Finance, 2009). Much of the investment goes to the development of teachers who are considered pivotal to the successful implementation of curriculum reforms under the Thinking School, Learning Nation (TSLN) vision of the Ministry of Education (MOE). At the recent annual Workplan Seminar where the MOE provides the direction for strategic planning by every sector in the educational system, there was a focus on teachers as "the heart of quality education" (Ng, 2009). The Minister of Education, Dr Ng Eng Hen announced at the seminar: the bumper harvest for new teachers, a new career track for Allied Educators hired to support the work of teachers, an increase of funds for schools to hire more adjunct teachers, more pathways for the upgrading of teachers, the development of a Teacher Development Centre (TDC), a new Superscale-grade Principal Master Teacher (PMTT) position as the apex of the Teaching Track and the creation of a new position of a lead teacher in schools. These announcements augur well for the teaching profession as the system works towards a target of some 33,000 teachers by 2015.

Recruiting Teachers

Teaching remains a highly desirable profession in Singapore. According to a public perception survey commissioned by MOE, teaching was viewed by the public as the most respected profession in terms of its contribution to society. Tertiary students

[1] A/P Christine Kim-Eng Lee is currently the Head of Curriculum, Teaching and Learning Academic Group (CTL), National Institute of Education, Nanyang Technological University, Singapore. She can be contacted at christine.lee@nie.edu.sg.
[2] Ms Mei Ying Tan is a research assistant in CTL and is currently pursuing her doctoral studies at Teachers College, Columbia University.

ranked teachers 2nd highest, next to doctors, and above other professions such as law, banking, and nursing (Shanmugaratnam, 2006).

Rating and rewarding teachers begins long before recruits become qualified teachers. To qualify for an interview, applicants need to be within the top 30% of their cohort, should have relevant higher education, pass tests in literacy, and show evidence of interest in educating children. Once they make the paper review cut, applicants would have to undergo an interview process to determine their suitability to be a teacher and a role model for their students. The interview focuses on whether the applicant has strong communication and interpersonal skills, a willingness to learn and a strong motivation to teach (Sclafani, 2008). After a successful interview, they will enter the National Institute of Education and continue to be monitored especially during the practicum and will be asked to leave the profession if their performance is not satisfactory. Only one in five applicants enter the National Institute of Education, and out of these twenty percent, 18 percent eventually graduate to become teachers (McKinsey, 2007). To attract new people into teaching, Singapore provides good pay at the start –"frontloaded compensation" (McKinsey, 2007, p. 22) – paying a salary as well as the tuition fees of student teachers. Trainee teachers with degree qualifications (GEO 1.2 Untrained) are provided with monthly gross salaries ranging from S$2550 to S$2990. Meanwhile, trainee teachers without degree qualifications (GEO 2.2 Untrained) are paid gross salaries of S$1480 to S$1870 per month (NIE, 2010). Singapore is probably one of the few economies in the world that pays salaries to **all** pre-service teachers undergoing teacher preparation programmes at the National Institute of Education. Starting salaries for teachers upon graduation from NIE are comparable to other professions. The status of teachers is maintained at a high level, thus making teaching a desirable career choice and continues to attract more top students into the profession.

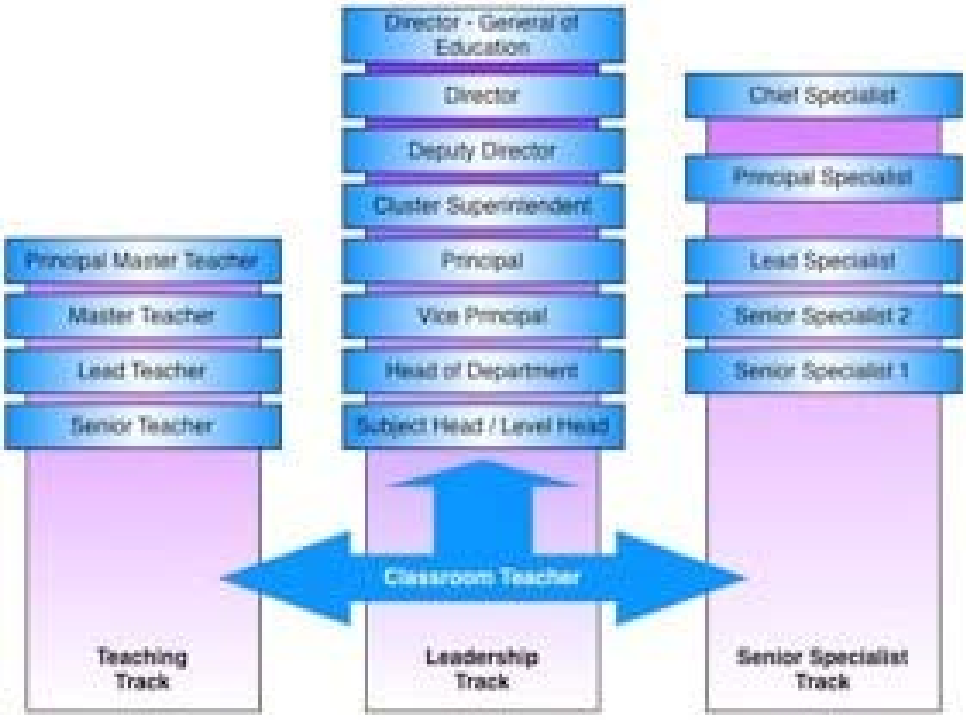
Rating Teachers

Appraisal and feedback have a strong positive influence on teachers and their work. In an OECD study of teaching appraisal and feedback and its impact on schools and teachers, teachers report that "it increases their job satisfaction and to, some degree their job security, and it significantly increases their development as teachers" (OECD, 2009).

Singapore pays a great deal of attention to the development of teachers through the conceptualization and implementation of a performance management system called Enhanced Performance Management System (EPMS) which was fully implemented in 2005. EPMS is part of the career and recognition system under the "**Edu**cation Service **P**rofessional Development **a**nd **C**areer Plan" (Edu-Pac) for teachers to develop their potential to the fullest (Teo, 2001). This structure has three components: a career path, recognition through monetary rewards, and an evaluation system. EduPac takes cognizance that teachers have different aspirations and provides for 3 career tracks for teachers in Singapore: the **Teaching Track** that allows teachers to remain in the classroom and advance to a new pinnacle level of a Master Teacher; the **Leadership Track** that provides opportunity for teachers to take on leadership positions in schools and the Ministry's headquarters and the **Senior Specialist Track** where teachers join

Ministry's headquarters and become a "strong core of specialists with deep knowledge and skills in specific areas in education that will break new ground and keep Singapore at the leading edge" (Teo, 2001) (see Fig. 1 for the levels within each of the tracks).

Fig. 1: Differentiated Career Tracks under EDU-PAC

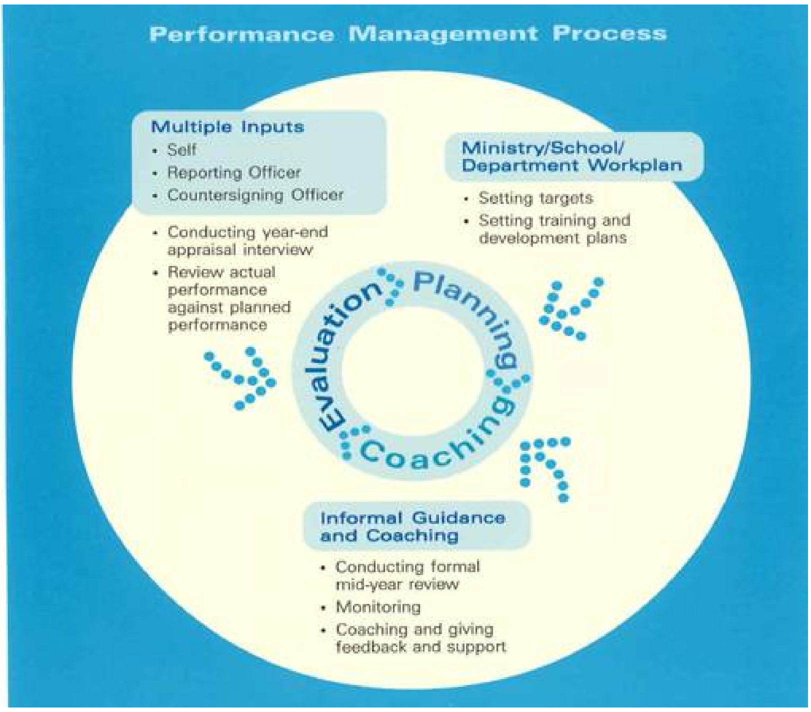


Source: MOE Website on Teaching as a Career

The Enhanced Performance Management System (EPMS)

The EPMS (Ministry of Education, 2005) has 2 aspects: (i) it is competency-based and defines the knowledge, skills and professional characteristics appropriate for each track thus providing clarity in terms of the expectations and behaviours needed for success in each of the tracks; and (ii) it is developmental in nature and supports teacher improvement and performance. The process involves performance planning, performance coaching and performance evaluation. In **performance planning**, the teacher starts the year with self-assessment and develops goals for teaching, instructional innovations and improvements at the school, professional development and personal development and meets with his/her reporting officer who is usually the Head of Department for a discussion about target setting and performance benchmarks. **Performance coaching** takes place throughout the year and more so during a formal mid-year review where the reporting officer meets with the teacher to discuss progress and share needs and to coach and provide feedback and support. In **performance evaluation** held at the end of the year, the reporting officer conducts the appraisal interview and review actual performance against planned performance (see Figure 2). A performance grade is given and this would affect the annual performance bonus received for the year's work. It is also during the performance evaluation phase that decisions regarding promotions to the next level are made based on "current estimated potential (CEP)". The decision on a teacher's current estimated potential (CEP) is made in consultation with senior staff who has worked with the teacher, based on "observations, discussions with the teacher, evidence of portfolio, and knowledge of the teacher's contribution to the school and community" (Sclafani, 2008).

Fig. 2: Performance Management Process



Source: MOE, Enhanced Performance Management System –
Teaching Fields of Excellence, p.7

Teachers are evaluated based on the "what" and "how" of performance. The "what" of performance is captured through Key Result Areas (KRA) which describes the broad areas of work expected of a teacher. The KRAs for the Teaching Track are:

- The holistic development of students through:
  - Quality learning of students
  - Pastoral care and well-being of students
  - Co-curricular activities
- Contribution to the school
- Collaboration with parents
- Professional development

The knowledge and skills expected of a teacher complements the KRAs and they are:

- Knowledge:

- Teaching Area – the content and curriculum knowledge that teachers must know to teach in the classroom
- Psychology – the knowledge of child development that teachers must know to maximise pupil potential.
- Developments in the field of education – the knowledge in other areas of education that mould a complete educator
- Education Policies – the rationale and the philosophy that sets the direction and focus for teachers to carry out their tasks.

- Skills:
  - Teaching Pedagogy – the pedagogic techniques and approaches that teachers must practise to teach in the classroom.

Teachers are assessed based on their competencies which capture the "how" of performance. There are 4 points on the rating scale, ranging from 'Not Observed', 'Developing' to 'Competent' and 'Exceeding'. According to the EPMS Dictionary, competencies are "the underlying characteristics that are proven to drive outstanding performance in a specific job and are the personal attributes and behaviours that lead to longer-term achievement and success".

There is a set of 13 competencies in the Teaching Competency Model which teachers can use to identify their strengths and weaknesses as areas of continuous learning and professional growth (see Fig 3). Teachers are assessed on 9 of the 13 competencies – Nurturing the Whole Child and the competencies related to Cultivating Knowledge, Winning Hearts and Minds and Working with Others. The remaining 4 competencies related to 'Knowing Self and Others' are not used for assessment purposes but are considered as emotional intelligence competencies important for self development.

Fig 3: Teaching Competency Model

Source: MOE. Enhanced Performance Management System –
Teaching Fields of Excellence, p.4

An important core competency in the teaching track is the ability of teachers to 'nurture the whole child'. Teachers are rated on increasing levels of ability to nurture their students holistically as shown below.

Level 1: Shares values Shares values with the child through advice, feedback and discussions, with the intent to nurture the whole child.
Level 2: Takes actions (GEO 1/2/ GEO1A1/GEO2A1/ GEO1A2/GEO2A2/ GEO1A3/GEO2A3 Sees the possibilities in each child and takes appropriate actions to convince him of values, and improve his self-confidence.
Level 3: Strives for the best possible provision (Senior Teacher) Acts consistently in the interest of the child and persists in working for the best possible outcomes.
Level 4: Encourages others to act in the best interest of the child (Master Teacher level 1) Encourages others in the school community to participate in the educational process to realise the child's full potential.
Level 5: Influence policies, programme and procedures (Master Teacher level 2) Takes an active role in initiatives that influence policies, programme and procedures in line with Nurturing the Whole Child

Teachers at higher stages of their career are expected to perform at higher levels, for example, a Master Teacher is expected to perform at levels 4 or 5 whereas a Senior Teacher is expected to perform at level 3. The table below shows behavioural indicators for a Senior Teacher and a Master Teacher for the core competency of nurturing the whole child.

Level 3: Strives for the best possible provision (Senior Teachers) Acts consistently in the interest of the child and persists in working for the best possible outcomes.	**Level 4: Encourages others to act in the best interest of the child (Master Teacher level 1)** Encourages others in the school community to participate in the educational process to realise the child's full potential
Behaviour Indicators Seeks out opportunities to stretch the students' abilities and maximise their potential Willing to do what is necessary to help students overcome challenges. Works in partnership with parents, relevant individual or authorities in the interest of students.	**Behaviour Indicators** Garners support of colleagues to support student to achieve potential Conducts workshops or sharing sessions with stakeholders to inspire further development of students Conducts studies to determine students' interests and abilities, and seeks to implement practices that benefit students' development. Leads others in the planning and implementation of projects that will benefit the students holistically

Subject Mastery is one competency under 'Cultivating Knowledge' and is defined as "the drive to find out more and stay abreast of developments in one's field of excellence". The levels and behaviour indicators for subject mastery are:

<table>
<tr><td colspan="2">Level 1: Has knowledge in subject area and awareness of educational issues (Classroom Teacher GEO1/2)
Shows keen interest in own subject area and related educational issues within subject area.</td></tr>
<tr><td colspan="2">Level 2: Keeps abreast with trends and developments in own subject area (Classroom Teacher GEO 1A1/2A1, 1A2/2A2, 1A3/2A3)
Takes initiative to stay current and expand content knowledge in own subject area</td></tr>
<tr><td>Behaviour Indicators</td><td>• Keeps updated on subject area through reading beyond curriculum requirements
• Has broad-based knowledge of subject area
• Actively seeks new information on subject area through attending courses, training etc
• Is proactive in offering to share expertise in subject area with colleagues</td></tr>
<tr><td colspan="2">Level 3: Applies knowledge of trends and developments into lessons (Senior Teacher)
Uses knowledge of trends and developments in own subject area in lessons.
Makes systematic effort over a period of time to obtain needed feedback or data to ensure effectiveness and relevance.</td></tr>
<tr><td colspan="2">Level 4: Develops innovative approaches (Master Teacher level 1)
Demonstrates deep understanding of current or new approaches to the future needs of the education system.
Develops approaches that could impact the education system.</td></tr>
<tr><td>Behaviour Indicators</td><td>• Uses tested approaches and strategies in subject area and introduces them to other colleagues at school/cluster level
• Develops new and creative strategies to deliver Lessons
• Designs an integrated curriculum and/or initiates a padagogical approach to meet the
• future needs of education</td></tr>
<tr><td colspan="2">Level 5: Provides thought leadership (Master Teacher level 2)
Explores and pushes the horizon in the one's subject area/teaching field</td></tr>
</table>

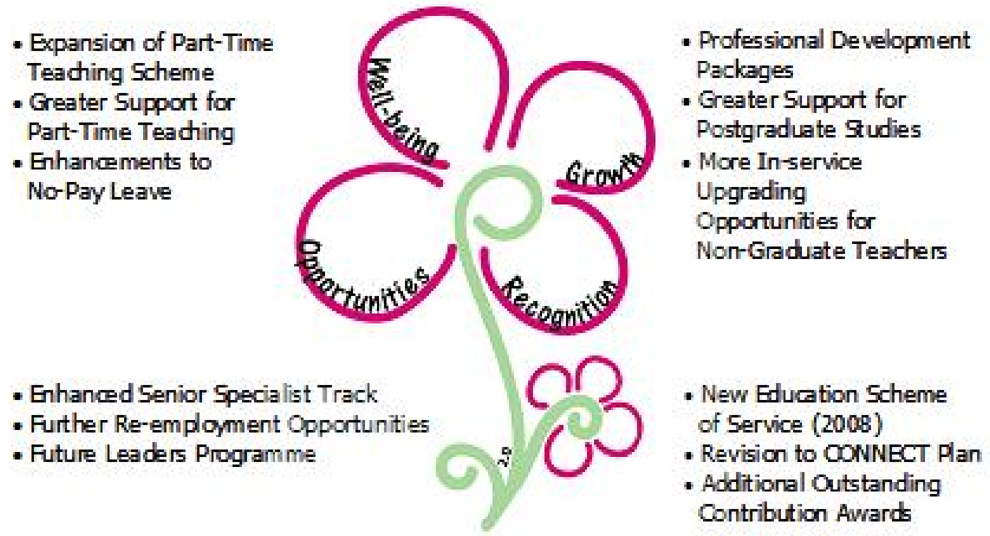
Rewarding Teacher Performance

Teachers are rewarded financially according to how they are evaluated in the Enhanced Performance Management System. In addition to the annual salary increments and based on their performance grades, they would be given a differential performance bonus which would amount to one to three months salary for average to outstanding performers, that is, a 10-30% annual bonus (Sclafani, 2008, p. 7). This performance bonus is awarded in March each year for the work done during January to December of the year before.

Grow 2.0 Package

An enhanced career structure, the GROW 2.0 package (**G**rowth of Education Officers, through better **R**ecognition, **O**pportunities, and seeing to their **W**ell-being) was introduced by MOE to "give teachers more recognition for excellence and commitment to their calling, more career options and professional development as well as greater flexibility in managing their career and personal lives". It is aimed at "the professional and personal development of teachers more comprehensively and holistically". Figure 4 shows the various dimensions of the GROW 2.0 package.

Figure 4: Dimensions of Grow 2.0 Package



Source: [MOE website on Teaching as a Career]

CONNECT Plan

An interesting dimension of the GROW package is the CONNECT plan (**CON**ti**N**uity, **E**xperience and **C**ommitment in **T**eaching) which was put in place to increase teacher retention and to reward teachers for staying on in the Education Service. The CONNECT plan encourages teachers to remain in the service until retirement by allocating a sum of $3,200 to $8,320 every year for each teacher and giving a monetary payout at defined points every 3 to 5 years ranging from $15,200 to $36,100. The payouts are higher in the first 20 years of service as teachers are expected to have higher financial commitments during that period. The total payout (payout quantum plus retention quantum) for each teacher (GEO 1/1A1/1A2/1A3/SEO levels) who stays in service for 30 years amounts to a princely sum of S$214,120.

Access to professional development (PD) opportunities

All teachers are also entitled to 100 hours of PD annually which can occur during school hours with resources provided for relief teachers. The Ministry also provides funding for scholarships and study leave—both locally and abroad and facilitates “teachers’ movement along selected career ladders and learning along multiple dimensions” (Goodwin, in press). For instance, MOE provides postgraduate scholarship for outstanding teachers tenable at top universities in the US, UK, Canada, Australia and elsewhere.

There is also a provision of enhanced sabbatical scheme for teachers so that they can take a break from teaching and do something else. Teachers with 12 years of service can take 2½ months of full-pay leave. This means they can take a full school term off, at full pay, to be on sabbaticals for a variety of purposes such as teach in a different type of school, pursue a higher degree programme locally or overseas, or go on a structured Teachers' Work Attachment in an organisation quite different from a school. (Shanmugaratnam, 2006).

Teachers could also claim up to S$400 or S$700 per year, depending on their years in service, for learning and development related expenses. This could be spent on the purchase of books, subscription to magazines and journals, purchase of personal productivity devices such as PDAs and webcams as well as payment for courses. Teachers are also rewarded with a sum ranging from $3,000 to $10,000, depending on staff strength for outstanding team-based contributions to their school. The individual outstanding contribution award is at S$1,000.

Given all the support available to new and continuing teachers, it comes as no surprise that the overall attrition rate due to retirement and resignation has remained steady at a low rate of 2.4%, a rate which MOE is committed to reducing (Goodwin, in press).

Conclusion

Salary and career advancement are constantly scrutinized to make rewards attractive to teachers. The reward system is focused on rewarding excellence and reinforces the idea that rewards equals contribution. The evaluation system through the use of EPMS reflects the totality of a teacher's contributions to their students, school and community (Sclafani, 2008). At the same time, continuous and deep support for teacher professional learning is a priority so that teachers would be able to perform well both in work and in year-end evaluations. The importance of professional development is in turn driven by strategic directions and priorities set by MOE (Wang-Iverson et.al, 2009). While the career and pay structures have been successful in attracting and retaining people, less had been studied about the implementation of EPMS and the extent to which schools are able to release teachers for professional development. It is necessary to conduct an honest evaluation of the effectiveness of EPMS and professional development from the teacher's point of view.

Singapore recognises that its system of evaluating teachers and the implementation of that system need improvement. In particular, the system is not evenly implemented across the school system (Lui, 2007, Siew, 2007). There is also a need to examine the domain-specificity of the evaluation system, for example, how does it apply to the assessment of math teachers. However, despite its imperfections, the system has helped Singapore attract people into the education service and retain them (Shanmugaratnum, 2006).

References

Goodwin, A. L. (in press). Quality teachers, Singapore style.

Lim, L. H. (2003). Impacting the positioning of teaching as life-long career. *Educational Research for Policy and Practice. 2,* pp. 87-92.

Lui, T. Y. (2007, March). *FY 2007 Committee of supply debate 6th reply by MOS Lui Tuck Yew on teachers workload and assessment of teachers*. Speech presented at Committee of supply debate, Singapore. Retrieved from http://www.moe.gov.sg/media/speeches/2007/sp20070307f.htm

McKinsey & Co. (2007, September). *How the world's best-performing school systems come out top*. Retrieved from http://www.mckinsey.com/App_Media/Reports/SSO/Worlds_School_Systems_Final.pdf

Ministry of Education. (2005). *Enhanced Performance Management System*. Singapore.

Ministry of Education, Singapore. (2010). *Career information*. Retrieved from http://www.moe.gov.sg/careers/teach/career-info/

National Institute of Education (2010). General Information handbook 2009 – 2010. Retrieved from http://www.nie.edu.sg/itt_hb/web/2010/general_handbook_Jan-Dec2010.pdf

Ng, E. H. (2009). *Speech by Dr Ng Eng Hen, Minister for Education and Second Minister for Defence, at the MOE Work Plan Seminar 2009, on Thursday, 17 September 2009 at 9.30am at the Ngee Ann Polytechnic Convention Centre.* Retrieved from http://www.moe.gov.sg/media/speeches/2009/09/17/work-plan-seminar.php

Sclafani, S. (with Lim, E.). (2008). *Rethinking human capital in education: Singapore as a model for teacher development.* A paper prepared for The Aspen Institute Education and Society Program. Retrieved from http://www.aspeninstitute.org/sites/default/files/content/docs/education%20and%20society%20program/SingaporeEDU.pdf

Shanmugaratnum, T. (2006, September). *Keynote address by Mr Tharman Shanmugaratnam, Minister for Education & Second Minister for Finance, at the teachers' mass lecture, 4 sep 2006.* Retrieved from http://www.moe.gov.sg/media/speeches/2006/sp20060904.htm

Siew, K. H. (2007, March 7). Of soldiers and teachers [Web log message]. Retrieved from http://siewkumhong.blogspot.com/2007/03/of-soldiers-and-teachers.html

Teo, C. H. (2001, January). *Speech by Radm (NS) Teo Chee Hean, Minister for Education and Second Minister for Defence at the Ministry of Education investiture of the 2000 national day awards on Saturday, 20 Jan 2001 at 10.00 am at the Singapore International Convention & Exhibition centre*. Retrieved from http://www.moe.gov.sg/media/speeches/2001/sp20012001.htm

Teo, C. H. (2001, July*). Speech by Radm (NS) Teo Chee Hean, minister for education and second minister for defence at the NIE teachers investiture ceremony on Wednesday, 4 July2001 at 9.30 AM at the Singapore indoor stadium.* Retrieved from http://www.moe.gov.sg/media/speeches/2001/sp04072001.htm

Wang-Iverson, P., Myers, P., & Lim, W. K. (2009). Beyond Singapore's mathematics textbooks: Focused and flexible supports for teaching and learning. *American Educator*, Winter 2009-2010, pp. 28 -38.

Fact Sheet about Teachers in Singapore

Teacher profile by grade level, academic qualification and age

MOE Teachers (all)

	Number & academic qualification				***Number of teachers at the ages of:***				
Grade level	**'O' level**	**'A' level**	**Under-grad**	**Post-grad**	***<24***	***25-34***	***35-44***	***45-54***	***55<***
Primary Grade 1-6 Total=13,023 (100%)	613 (4.7%)	4,840 (37.2%)	7,187 (55.2%)	383 (2.9%)	*461 (3.5)*	*6223 (47.8)*	*3,931 (30.2)*	*1,494 (11.5)*	*914 (7%)*
Secondary Grade 7-10 Total=12,143 (100%)	90 (0.7%)	875 (7.2%)	10,078 (83%)	1,100 (9.1%)	*420 (3.5)*	*5,702 (47%)*	*3,313 (27.3)*	*1,858 (15.3)*	*850 (7%)*
Junior College Grade 11-12 Total=2728 (100%)	2 (0.07%)	9 (0.33%)	2,227 (81.6%)	490 (18%)	*65 (2%)*	*1,473 (54%)*	*596 (22%)*	*403 (15%)*	*191 (7%)*

Comment: This does not include records for teachers in polytechnic and Institutes of Technical Education. Figures in brackets refer to percentages.

Source: MOE. *2009 Education Statistics Digest*

Adapting Lesson Study in APEC Member Economies

Maitree Inprasitha, Khon Kaen University

February 2010

Introduction

After the 1999 Educational Act has been launched, the first decade of educational reform movement (1999-2009) during the information age started in Thailand. In order to be aligned with the national agenda – "to reform learning process"- inaugurated in the act, new curriculum called "2544 B.E. Basic Education Curriculum" has been implemented in schools nationwide. Policy makers, curriculum developers, other related educational personnel, and teachers themselves seemed to notice the distinguished features of this new curriculum which emphasizes not only contents or subject matters but also learning processes, and desirable characters. During the first half of the first decade of educational reform, we witnessed many attempts to respond to the new curriculum demands, observing a huge number of innovative programs and projects implemented in schools with the support of the Ministry of Education, other governmental and non-governmental organizations, and even projects initiated by schools themselves.

In order to encourage and support teachers to contribute to the success of those attempts, the government has made every effort to contributing to this big educational reform. In particular, there is a new promotion system for teacher professional development which is correspondent to the promotion system in the university, that is, the teachers can be promoted to get both basic salary and position salary if their academic work has been approved by the ad hoc committee. This promotion system stimulated the teachers in every school to conduct their academic work, not only teaching work as they used to be done before, for example, doing some kinds of classroom research, documenting it and submitted to the ad hoc committee for approval. Unfortunately, this observed phenomenon can be called 'successful' in terms of promoting 'only' the teachers but not 'the students'. A number of newspaper headlines read *'has not much change in the classroom'*, *'still need innovation for real change in the classroom'* etc. It seems that classroom has not been changed. We have been struggling for 'Best practices" on how to change the classroom which really promote 'teachers' as well as students' learning reform in the classroom.' According to this background, many Thai educationalists are very concerned that maybe Thailand must begin a new journey and embark upon a second decade of educational reform (2010-2019).

Adapting Lesson Study in Thailand

The history of how Japanese Lesson Study has been adapted into school culture in Thailand started a decade ago. In the year 2002, Inprasitha (Inprasitha, 2008) implemented lesson study with 15 student teachers voluntarily in teaching mathematics in 7 secondary schools in Khon Kaen districts and nearby. In this first attempt, three phases of lesson study were implemented. The process of lesson study was adapted in the following manner.

In the planning phase, Inprasitha (2008) as a researcher, coached 15 students to spend a lot of time to carefully create lesson plans by emphasizing mathematical activities using open-ended problems. Half of the lesson plans to be used in the first semester were completed before they went to schools. Usually, student teachers create their lesson plans when they go to schools and only a week before they teach. In the implementation phase, the student teachers taught the lessons in the usual classrooms, observed at times by their friends teaching at the same school and who had to observe the classes in their spare time. They had to keep a diary or journal to record the problems of using mathematical activity with open-ended problems and the students' responses to the mathematical activity (e.g., how the students interpret the instruction of the activity, the vagueness of the instruction, expected and unexpected students' responses to the mathematical activity). In the post-discussion phase, since we could not conduct the discussions immediately after teaching classes, all student teachers came back to the Faculty of Education and spent at least 3 hours every Friday completing post-discussion with the coach.

Inprasitha (cf., Isoda et al., 2007) summarized the result of this project as follow:

During the first half of the semester all student teachers in the project experienced some difficulty adjusting to their new teaching roles and classroom organization. Participation in the weekly seminar facilitated the student teachers gradual change of the teachers' role. The most critical point of change was encountered while sharing their differing teaching experiences among friends and colleagues. Sharing experiences with their friends during the weekly seminar not only resolved their common concerns but also developed and expanded their own pedagogy, teaching practices and professional development. <u>The greatest paradigm shift for student teachers was that teaching mathematics does not mean focusing on the coverage of content but emphasizing the students' learning processes, original ideas, attitudes towards learning mathematics and satisfying one's own competence.</u> To scrutinize what adaptation we made in this project, two issues are 1) the integration of lesson study and Open Approach and 2) the outside person's roles in stimulating change in schools with profound understanding of how to improve sustainable professional development.

With the impression of success of this small project, the researcher attempted to expand implementing lesson study with in-service teachers and with more number of schools. Unfortunately, in the expansion of lesson study in 2004-2005, more focus was on using Open Approach and less emphasis on the process of lesson study. Although most of the teachers successfully used open-ended problems to change their classrooms and their roles in classroom organization and most of the students, especially those who rarely engage in mathematical activity, had more chance to actively access to mathematical activity, the great demand on open-ended problems is obvious. The teachers developed a sense that it is not the teachers who should create their own open-ended problems due to the limitation of time they have in schools. As a result, they request other organizations such as the center for research in mathematics education to provide open-ended problems so that they can smoothly handle the situation. This kind of attitude made the teachers reluctant to continue using Open-Approach. The researcher takes this risk by introducing the long-term project of how to implement lesson study in schools by incorporating Open Approach into Lesson Study process.

The three-year long project (2006-2008) implementing the integration of lesson study and Open Approach in 4 project schools started in the year 2006. More systematic adaption was conducted this time. The project was designed to have lesson study team including one university professor as a coach, at least two graduate students from master degree program in mathematics in each school, one school coordinator (new position working in this project) working four days at school and come back to the university for doing reflective seminar about the implementation of lesson study. School teachers, principals and supervisors also voluntarily joined this project in two of four schools. Lesson study in mathematics was conducted in 1st and 4th grades in 2006, 1st, 2nd and 4th, 5th grades in 2007, 1st, 2nd, 3rd and 4th, 5th, 6th grades in 2008. The project completed implementing lesson study in mathematics as a model for implementing lesson study mathematics in 9 years of compulsory education in elementary and expansion schools. These experiences have been sharing within the country and in APEC member economies through the project will be discussed in the next session.

Adapting Lesson Study in APEC Member Economies

1) Cycle of Implementing Lesson Study Member Economies

Among many attempts in responding to the national agenda and curriculum demand mentioned in the earlier session, there has been a modest initiative project to implement innovative teaching approaches to improve teaching mathematics and in turn improve professional development of mathematics teachers. ***"A Collaborative Study in Innovative Teaching and Learning Mathematics among Different Cultures in APEC member Economies"*** was proposed by Thailand and Japan to APEC HRDWG annual meeting in Pattaya, Thailand in 2005. An idea of how to implement lesson study in APEC member economies is as follows. The project overseers identified 'specialists' who have background in attempting to improve teaching mathematics and professional development in each economy. The project overseers intended that each specialist from a participating economy would have played a major role in implementing, expanding, creating lesson study network in his/her economy while sharing experiences with other specialists by participating in this project. A one year cycle of implementing lesson study in member economies is composed of 4 phases as below.

HRD 03/2006: A Collaborative Study on Innovations for Teaching and Learning Mathematics in Different Cultures among the APEC Member Economies

Phase I: An open symposium and a closed workshop for key mathematics educators from the APEC members economies on "Innovative Teaching Mathematics through Lesson Study" were held on January 15 – 20, 2006 in Tokyo by CRICED. The aim was to further refine a research proposal and a collaborative framework for the development of innovations and good practices for teaching and learning mathematics. At this gathering "Lesson Study" was selected as the key innovation.

Phase II: Based on the agreed collaborative framework, each of the cosponsoring APEC economies conducted a research during February and March 2006 in an actual classroom setting in their home economies to develop innovations and good practices in teaching and learning mathematics through Lesson Study.

Phase III: An APEC International Symposium on "Innovation and Good Practices for Teaching and Learning Mathematics through Lesson Study" was organized in Tokyo for the purpose of sharing and reflecting on research results and good practices as discovered by research teams of the economies. The Symposium was hosted by the CRME of Khon Kaen University, Thailand during June 14-27, 2006.

Phase IV: An APEC Workshop on "Improving the quality of the mathematics lesson through Lesson Study" was held in Thailand on August 24 – 27, 2006. Here, the Japanese teaching method was introduced to Thai teachers in the manner of a workshop on Lesson Study.

HRD 02/2007: Collaborative Studies on Innovations for Teaching and Learning Mathematics in Different Cultures (II) – Lesson Study focusing on Mathematical Thinking –

Phase I, Activities in the first phase are as follows: Lectures and a panel for sharing ideas of mathematical thinking to help develop lessons by teachers; and a Workshop to develop a collaborative framework for using Lesson Study to develop mathematical thinking. Specialists observed four research lessons in Japanese classrooms and shared the ideas of Lesson study to develop mathematical thinking. A forum where specialists shared their ideas on mathematical thinking based on the keynote lectures and their experiences, was held on December 2 – 7, 2006 in Tokyo & Sapporo, Japan.

Phase II, Each co-sponsoring APEC member economy engaged in the Lesson Study project for developing some topics on communication (February-July 2008).

Phase III, An International Symposium and a Lesson Study meeting (a kind of workshop for teachers) was organized in order to share teaching approaches for developing communication by economies. The symposium was hosted by Center for Research in Mathematics Education (CRME), Faculty of Education, Khon Kaen University, Thailand (at Khon Kaen, August 16 – 20, 2007).

Phase IV, A workshop in Khon Kaen, Thailand (August 15 – 16, 2007)

HRD 02/2008: Collaborative Studies on Innovations for Teaching and Learning Mathematics in Different Cultures (III) – Lesson Study focusing on Mathematical Communication.

Phase I, A workshop and a lesson study meeting which is designed for specialists among key mathematics educators from APEC member economies hosted by Center for Research on International Cooperation in Educational Development (CRICED), University of Tsukuba, Japan was organized in order to share the ideas and ways of communication on curriculum level and teaching level (at Tokyo & Kanazawa, December 2007).

Phase II, Each co-sponsoring APEC member economy engaged in the Lesson Study project for developing some topics on communication (February-July 2008).

Phase III, An International Symposium and a Lesson Study meeting (a kind of workshop for teachers) was organized in order to share teaching approaches for developing communication by economies. The symposium was hosted by Center for Research in Mathematics Education (CRME), Faculty of Education, Khon Kaen University, Thailand (at Khon Kaen, August 24 – 28, 2008).

Phase IV-1, Lesson Study as a theme in 11th ICME in Monterrey, Mexico, July 6 – 13, 2008

Phase IV-2, First Open Class of Lesson Study in Australia in International Conference in Sydney in March 16 – 18, 2009)

Summary of the Results of 3 Years Project (2006-2008)

Based on the procedure mentioned above, the results obtained during the four phases are presented below.

Developing innovations by focusing on good practices

1) **Clarifying the meaning of good practices**

At the beginning of this project, specialists/project overseers tentatively proposed the meanings of good practices in mathematics education with the following conditions:

1) Good practices must be visible and can be recorded in the classroom and demonstrated to other people.
2) The practices must be accepted as a good practice in each member economy.
3) There must be at least one teacher who is well known for that approach.
4) The practices must be useful for mathematics education reform in each member economy.
5) The practices must cause other teachers to wish to apply the same approach to their teaching.
6) The practices must be known as being useful for teacher education (pre-service and in-service alike).
7) Comparatively, there are different/traditional approaches based on different/traditional values.

To collaboratively develop innovations in teaching and learning mathematics, the project focused on gathering good practices using video recording from specialists in each economy and discussed “what is good,’ ‘why is it,’ and ‘how can the teachers develop such a good practice.’ Watching the specialists elaborate on these themes through the videos, we were able to observe how each of these ‘good practices’ had been developed in different cultural settings. Based on these differences we could further re-evaluate our own ‘good practices’ from a new/different perspectives. We also found new ideas and teaching methods which can be applied to mathematics classrooms in APEC member economies.

2) **Deciding to use Lesson Study as a means of developing good practices**

As in phase I, specialists attended the conference with many lectures given by keynote speakers on Lesson Study. They then observed four lesson study classes at the attached schools to the University of Tsukuba in Tokyo. They also presented a variety of ideas on good practices through their papers or complementary videos. Japanese Lesson Study originated in Japanese culture. While specialists in some economies experienced its adaption in their local school contexts, specialists in other economies may not be aware of it. The participants elaborated and shared the significance and meanings of Lesson Study obtained from those experiences and to participate in the Lesson Study organized at the attached schools. Eventually, all the participants reached a consensus on the application of Lesson Study as a means to develop good practices in the economies. The responses to the questionnaires distributed at the meeting in this regard were very positive.

The process of developing and sharing good practices

In Phase II, based on the shared idea on using Lesson Study as a collaborative framework, each of the specialists developed his/her innovations by emphasizing good practices in teaching mathematics.

Also in Phase II, specialists from each participating economy produced videos on good practices during the conduct of their research activities to share with their colleagues at the International Symposium in phase III.

In Phase III, recognizing different cultures as an attribution to innovative idea, specialists found various challenges (e.g., the adaptation of Lesson Study) to develop good practices in teaching mathematics. The specialists also learned from each other about how to meet these challenges.

In Phase IV, some specialists produced videos on good practices from the workshop in Thailand and gathered some videos on good practices from other Lesson Study project such as the one in Chile.

Implementing Lesson Study in member economies

1) Experimenting Lesson Study

In Phase II, specialists began to use Lesson Study in each economy to exemplify the meaning of good practices. Specialists from thirteen economies engaged in the Lesson Study activities and subsequently nine videos on good practices were produced.

2) Sharing the results of Lesson Study through the prepared videos

In Phase III, specialists observed the local mathematics classes and observed classes conducted by a Japanese teacher. In this phase, they also reported on the findings of each research project conducted in Phase II. They then shared the methods they had observed in an attempt to describe good practices through video recording. The responses from the audiences to the questionnaires distributed at the meeting regarding research findings were very positive.

3) Producing videos on Lesson Study for teacher education

In Phases III and IV, specialists tried to expand Lesson Study for the benefits of in-service teachers by organizing a Lesson Study workshop for local school teachers in their respective economies such as Chile and Thailand. At these activities, six Lesson Study videos were produced to be used as a model of good practices.

Summary on Implementing Lesson Study in APEC Member Economies

In 2006, lesson study has been introduced in 12 APEC member economies as an innovative method for teaching and learning mathematics. With different school cultures, each economy has learned a lot of the problems when we acculturate lesson study as a cultural activity in school. In 2007, as lesson study became a part of school culture, the project focused on how to develop student's mathematical thinking in classroom. In 2008, after the specialists met in Tokyo and Kanazawa and observed how Japanese teachers develop their student's mathematical communication, they challenged schools in their economies.

As we have done in the last two years, by using videos, we, specialists of the project, shared our ideas on "good practices" in Khon Kaen, Thailand in August 2008. We also collaboratively reflected upon what we have done in the collaborative framework. We decided to keep and continue using this collaborative framework and to focus more deeply on the specific themes we think are necessary for school mathematics in each economy.

Besides, this project encourages each participating economy to expand this collaborative framework – Lesson Study – in each economy, in order for Lesson Study to be gradually integrated in school cultures of that economy. For example, as a case of success, in Thailand in 2009 the Ministry of Education set an educational policy to expand lesson study in 12 provinces and will expand more in the next three years countrywide. At the same time, Australia first started a public open class in the Sydney Conference March 2009.

As 2008 passed, the project concretely produced "good practices" such as classroom videos, progress reports, proceedings. These appeared both in hard copy and on the website of related organizations (e.g., those of CRICED (www.criced.tsukuba.ac.jp/math/apec2006), CRME (www.crme.kku.ac.th), and HRDWG Knowledge Bank wiki (http://hrd.apecwiki.org/index.php/Classroom_Innovations_through_Lesson_Study).

More importantly, the project created networks among mathematics educators and teachers, inside and outside of APEC member economies.

In the year 2009, under the economic crisis, the project overseers decided to extend the project by organizing 2009 Sydney Conference and this event included the first open class in Australia. After APEC Lesson Study strengthened collaboration among specialists and began to facilitate teachers networking at the national and international levels, the 4th APEC Lesson Study has shifted its attention from focusing on mathematical learning process to more cross-cutting themes like 'assessment' and also extended their interest to engage in the discipline of science. The new three-year proposal focusing on teacher education was submitted at the APEC HRDWG conference in Hiroshima during Feb 25-28, 2010.

References

Inprasitha, M. et all. (2003). *Reforming of the Learning Processes in School Mathematics with Emphasizing on Mathematical Processes*. Khon Kaen Printing.

Inprasitha, M. & Isoda, M. (2006). *Evaluation Report of a Collaborative Study on Innovation for Teaching and Learning Mathematics in Different Cultures among APEC Member Economies.*

Inprasitha, M. & Isoda, M. (2008). *Proceeding of APEC-Khon Kaen International*

Symposium 2008, Innovative Teaching Mathematics through Lesson Study III – Focus on Mathematical Communication –.

Inprasitha, M. & Isoda, M. (2009). *Collaborative Study on Innovation for Teaching and Learning Mathematics in Different Cultures (III) – Lesson Study focusing on Mathematical Communication -.*

Inprasitha, M., Loipha, S., Suttiamporn, W. (2008). Status quo and the subject of Mathematics Education in Thailand. In Saito, N. *Japan-Thailand International Seminar in Mathematics Education*. pp.5-16.

Inprasitha, N. (2009). *Lesson Study: An Innovation for Teacher and Student Development*. Doctor of Education Thesis in Curriculum and Instruction, Graduate School, Khon Kaen University.

Isoda, M., et all. (2006). *Tsukuba Journal of Educational Study in Mathematics.* Vol.25.

Isoda, M., Stephens, M., Ohara, Y., Miyakawa, T. (2007). *Japanese Lesson Study in Mathematics.* Singapore: World Scientific.

Lewis, C.C., (2002). *Lesson Study: A Handbook of Teacher-Led Instructional Change*. RBS, Philadelphia.

Wang-Iverson, P., & Yoshida, M., (2005). *Building Our Understanding of Lesson Study*. RBS, Philadelphia.

Lesson Study: Japanese Problem Solving Approaches

Masami Isoda
Center for Research on International Cooperation in Educational Development/
Graduate School of Human Comprehensive Science
University of Tsukuba, Japan.

February 2010

What is the lesson study and Japanese Approach?

Figure 1. Teacher's Canon (1873)

An Origin: Japanese lesson study School (University of Tsukuba) and the Attached Elementary School (Elementary School Attached to the University of Tsukuba) were established at the same time (Isoda et al 2007). It began from the observation of teaching methods in whole classroom teaching which were firstly introduced in those Schools beyond the temple school culture on tutorial teaching methods. People observed the ways of teaching for knowing how-to. Teachers' Canon was published by the Normal School in 1873 which already mentioned the etiquette for entering classroom for observation as for avoiding troubles during observations,

The conditions: There are various understanding of lesson study. Here, the Japanese lesson study is recognized with following features.

Process: Plan (Preparations), Do (Observations) and See (Discussion and Reflection) activities (lesson study cycle) with other teachers.

Various dimensions of observation:
Personal, whole School, regional and national but systematic

Theme: Study Topics and Objective
Study Topics such as Developing Mathematical Thinking, Learning for/by themselves in relation to development, reform or improvement.
Objectives related with curriculum, such as 'Through A, Teach B'. Both learning how-to (A) and achievement (B) are objectives of curriculum.

Lesson Plan: A format is usually developed/improved depending on a study topic. Some countries recommend a set of national lesson plans as a part of curriculum but lesson study is implemented for new challenges and not necessary to keep the same lesson plan.

Teachers' mind: Lesson study is conducted by teachers for developing students in a classroom and making each student developing him/herself. Not for researchers who just observe a classroom through their telescopes prohibiting influence into the classroom and do not feel sympathy teacher's objectives and do not consider next teaching activity in each moment. In this sense, lesson study recommends that researchers are teachers who propose improvement of class, as well as teachers are researchers who analyze children's understanding.

Result:

Lesson study usually considers achievement in relation to study topic and objective. At the same time, aims of lesson study change depending on participants and are not always the same as seen below; Model teaching approach, New ideas for traditional approach, Understanding objectives, What students learned before the class, What students learned and could not learn in the class, Teachers' values, Students' values, Professional development,…., Ideas for the curriculum reform, Theory of mathematics education, and so on. One of the most sharable products is a description of model approaches. The guidebooks for teaching contents and teaching approaches have been written by teachers. In these twenty years, videos have been used for sharing good approaches by making them more visible (APEC lesson study project 2006: first announcement). In some countries, a model approach sometimes means a teaching manual with the sequence of teacher's questions and children's answers which are expected to be followed by every teacher. But in the case of lesson study, it is nature of that to work beyond a model because lesson study usually includes a proposal to develop something new in their group based on their own theme of lesson. Thus, on the context of lesson study, a model approach means an illuminating approach and major resources for adapting a model into each teacher's classroom. And sometimes it means an object of improvement for specific aims.

<u>Developing Students thinking and learning by/for themselves</u>: Sometimes, general educators and educational management researchers enhance the function of the professional development on the lesson study but do not concern preparation of subject matter and teaching approaches for improvement. If it does not have the subject and teachers'

perspective for developing children, it is not satisfying the meaning of lesson study. The history of lesson study has been described with a new theme and a new approach on lesson study for developing children because the new theme and approach themselves are the aims of study and represent the reform, improvement, or focus of study itself.

The first known lesson study guidebook for teachers in Japan which have these features is ‘Reform the Methods of Teaching’ (1883: see figure 2). The lesson study topic was Pestalozzi methodology of teaching approach for whole subjects but it was not same as the original version in German because it was imported from New York Oswego Normal School and adapted in the Japanese way. In those days, lesson study had been introduced in Japan in a top-down way as well as establishment of the school system with an initiative of the government.

Another important feature of the first guidebook is the establishment of model teaching approach through questioning (‘Hatsumon’, as we call it today) *for developing students who think by themselves*. For enhancing a dialogue style of classroom communication in whole classroom teaching, the model approach itself was described through the dialogues such as ones of Plato and Confucius. The model dialogues in order to represent the process within a limited number of pages at high cost of publication are a recommended process for enabling teachers to plan their lesson and did not develop for following the protocol to describe social phenomena by current researchers on social science. Teachers’ guidebooks in Japan have been keeping the custom of the model dialog because it is much reproductive than the social-science-like protocol. From the viewpoint of teachers who are trying to reproduce his approach based on the model approach, model dialogue description style is reasonable because careful protocol as for data only describes the past as the object of interpretation and does not aim for designing new practice.

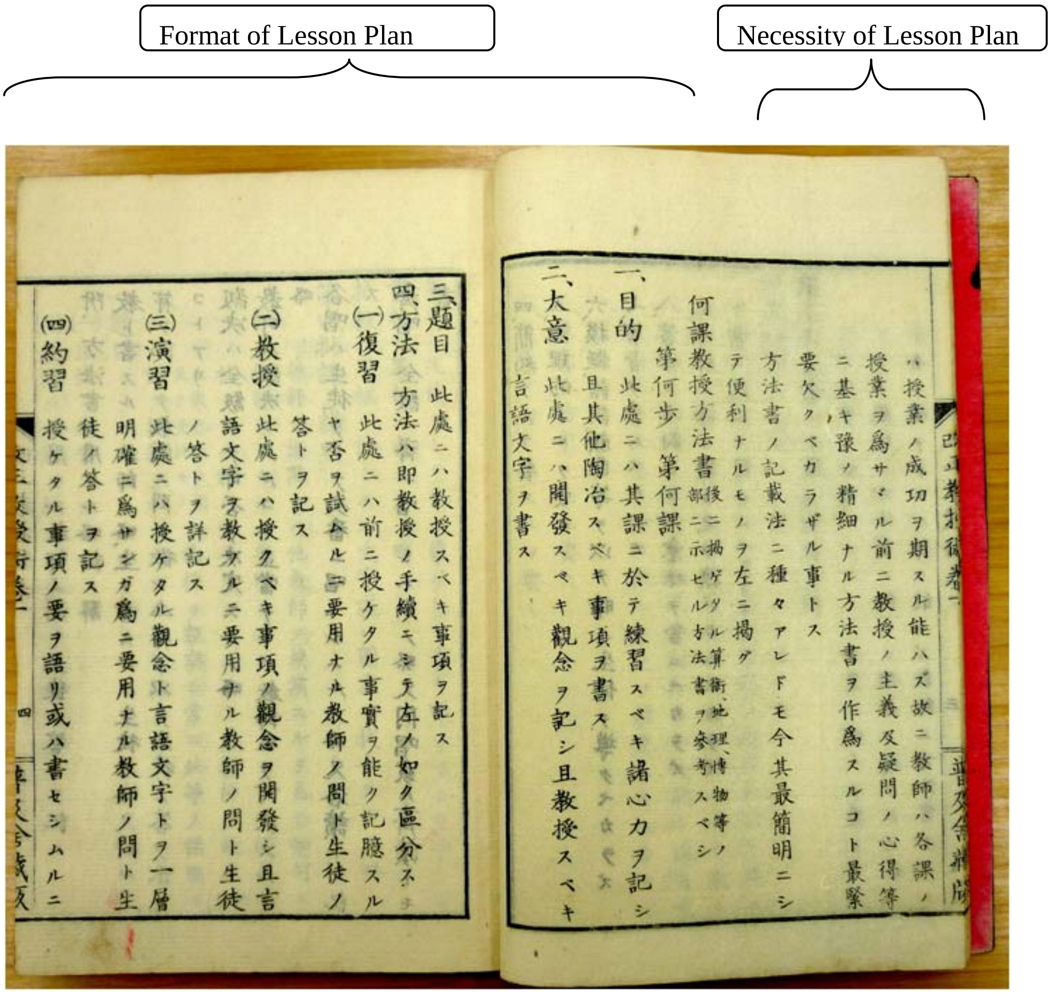


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Figure 2. ‘Reform the Methods of Teaching’ (1883)

In early 20th century, the bottom up way of innovative teaching approach movements appeared and influenced by movements of educational reforms in Germany and the U.S. Jingo Shimizu wrote a book ‘Teaching Elementary School Mathematics through Problem Posing by Children’ (1924) which explained the innovative teaching approach including a fact that an activity of learning mathematics begins from children’s problem posing. In this era, *Japanese Teaching Principle, ‘Learning by/for Themselves’* had been described by teachers and educators who wrote the teachers guidebook for teaching.

Since the end of World War II, developing thinking ability by themselves and learning by/for themselves have been major issues of the national curriculum standards. Problem Solving Approaches became a major method of teaching approaches (Isoda et al 2007). The origin of it was before WWII, but it has spread in 1980s and became standards approach in 1990s.

Figure 3. Problem Posing Approach by Jingo Shimizu (1924)

<u>Problem Solving Approach for Leaning by/for themselves</u>: Japanese Problem Solving Approach, known as the process through ‘posing a problem’, ‘independent solving’, ‘comparison and discussion’, and ‘summary and application’, was known in the US through the comparative study on problem solving in the 80s by Tatsuro Miwa and Jerry Becker. It influenced the world through the TIMSS video study in 90s (Stigler and Hiebert 1999). Problem Solving Approaches are one of the shared approaches in Japan and developing such a sharable approach itself is one of the long-term results of lesson study. Lesson study spread into the world with Problem Solving Approach. It may not have been spread if it were only explained by the lesson study cycle. The problem solving approaches combined with lesson study has spread to the world from Japan through the comparative studies and teacher training programs for developing countries from 1980s, the Japan International Cooperation Agency’s projects from 1993 (See, Isoda et al 2007) and APEC projects from 2006.

What are the lesson study strategies for developing children at School level?

Each Japanese elementary school usually sets a theme of lesson study project on school level through a year depending on the demands of national reform movements, teachers and school district. Major themes of lesson study projects at elementary schools are Japanese, Mathematics or general topics. General topics are usually related with crossing curriculum topic such as Physical and Mental health. More than 50 years, improvement of mathematics teaching for better achievement of curriculum has been a major theme of lesson study (Isoda et al 2007). In these days, the achievement of Japanese, Mathematics and Science on PISA has been lower due to the 20 % reduced curriculum in 1999, Mathematics and Japanese are two major subjects in Elementary School lesson study project

On this context, more than 50 teaching guidebooks for elementary school mathematics are published, every year. Here, for explaining about lesson designing strategy and showing how meaningful it is for improvement of children' performance, the teachers' guidebook titled 'Designing Problem Solving Class with the Basic Standards for Teaching Given by Check Sheets' by Isoda (2009), is introduced because it is currently known as one of the best-sellers in this area: the 1st printing was sold out within two months and now the 2nd printing is selling and also some of the same checking sheets are already published in Spanish (Isoda and Olfos 2009).

A characteristic feature of this book is that it is written as the result of school level lesson study project and descried for novice teachers who do not know well how to teach mathematics even if they might have several years of experiences.

In Japan, problem solving approaches are shared to develop children's ability to think and learn by themselves. For knowing their achievement, there are two sets of assessment tests problems in the national assessment. First type focuses on understanding and skills and second type focuses on mathematical thinking including mathematical argumentation. Both tests problems are developed on the national curriculum standards and the problems of the second type are deeply related with problems solving approach itself.

<u>The Checklists as for Strategy in school level lesson study</u>: The book by Isoda includes several checklists for teachers to develop teachers' pedagogical content knowledge and for children to develop their knowledge for learning how to learn. The lists are developed by the core teachers group in Ozone Elementary School in Tsukuba City with Isoda to improve teachers' pedagogical content knowledge and children's achievement through lesson study on problem solving approach. The following checking list (figure 4) is an example of teacher's checklists for lesson planning which are used by self-evaluation before every lesson observation and used after the every observation for knowing the reflection points:

Problem Posing	**Self-Evaluation**
1. The lesson sets tasks that can be solved in a variety of different ways by applying previously learned knowledge, and presents the content to be learned.	4 3 2 1
2. The lesson planned with tasks (problem given by teacher) and problems (problematic from students), and promotes problem (problematic) awareness.	4 3 2 1
3. The teacher expected methods and solutions before.	4 3 2 1
Independent Solving	
1. The children can recall and apply what they have already learned.	4 3 2 1
2. The children's ideas are predicted before.	4 3 2 1
3. Inappropriate solutions are predicted, and advice and hints are prepared for them before.	4 3 2 1
4. The teacher, walking around, observes and helps children to	4 3 2 1

insure that children use mathematical representation to solve the problems.				
5. Notebook are written and taken in a manner such that they will be helpful for presentation as well.	4	3	2	1
Comparison and Discussion				
1. Steps (Validity, Compare, Similarity and Generalization or Selection) are planned for comparative discussion.	4	3	2	1
2. The ideas to be taken up are presented in an order that is planned before.	4	3	2	1
3. The method for writing presentation sheets is planned in advance and directions are provided.	4	3	2	1
4. In addition to develop the ability to explain, children are also fostered with the ability to listen and the ability to question.	4	3	2	1
5. When ideas are brought together (generalized), it is important to experience them by themselves.	4	3	2	1
6. The reorganization or integration of ideas proceeds smoothly from the presentation and communication of children.	4	3	2	1
Summary				
1. Activities are incorporated that let children experience for themselves the merits of the ideas and procedures that are generalized.	4	3	2	1
2. The summary matches the aims and problems (problematic) of this lesson.	4	3	2	1
3. It is recognized that both correct and incorrect answers (to the task) have something good in the foundation of their ideas.	4	3	2	1
4. Children are made to experience the joy and wonder of learning.	4	3	2	1

Figure 4. Lesson Planning Checklist: Self-Evaluation [4: Achieved; 1: Not Achieved]

Isoda (2009), Isoda & Olfos (2009)

On this lists, the deference of problem (task) and problematic (problem) is a key because problematic is necessary for children leaning by/for themselves. On the other hands when the school began to use the checking lists on their project, most teachers did not understand the meaning of each checking list. Because in the case of this school most of teachers do not know how to teach mathematics well even if they have a chance to see other teacher's problem solving approach. After conducting school-level lesson study project for a year and half, through having lesson study once a month in each grade, the teachers well understood the meaning of check lists and developed high achievement.

<u>The Achievement of the School Level Lesson Study</u>: After the one and half year mathematics lesson study project in Ozone Elementary School through using checking lists for mathematics, children's achievement improved as follows.

In figure 5, Children's mathematical thinking ability which is a key for leaning by/for themselves is improved. It shows that achievement of children in the 5th grade improved by 15 points in mathematical thinking test compared with the average of the whole prefecture. Figure 6 implies that the achievement of school-level mathematics lesson study during one and half year is not only limited to the improvement of children's mathematics achievement but also influenced positively other subjects such as Japanese, Science and Social Studies. It means that the lesson study efforts on the teaching approach in mathematics through using checking lists may influence other subject of teaching. Indeed, in Ozone Elementary School, a teacher teaches almost all subjects. Children's awareness of empowerment in mathematics led to improvement of their interests of learning and developed their wish to study.

Figure 5. Ozone Elementary School's Academic Abilities Compared to the Regional Average

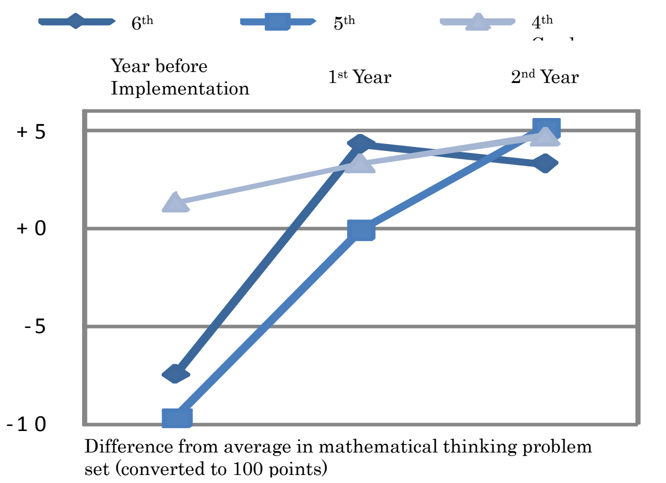
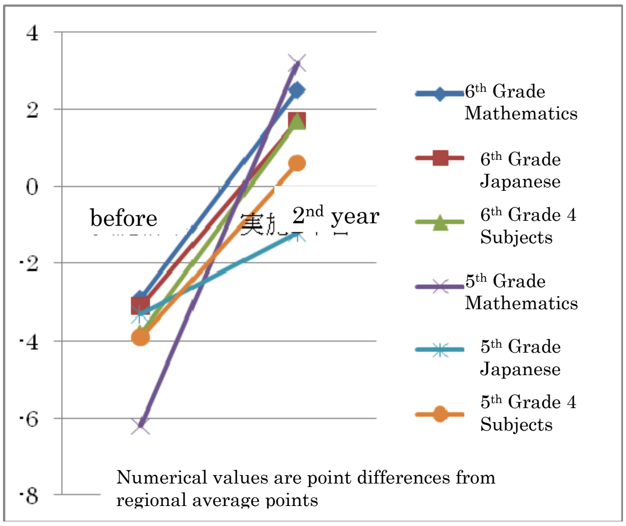


Figure 6. Ozone Elementary School's Academic Abilities Compared to the Regional Average



The achievement is the result of lesson study through using the check lists in the school. For improvement of classroom teaching, it is important that teachers and children share objectives. Ozone Elementary School developed Lesson Planning Checklist, Children Leaning How to Learn Checklist, and Lesson Plan Checklist, and also more checklists such as the way or blackboard planning were added in the book by Isoda (2009) and improved on Isoda and Olfos (2009) for Latin America. They provide opportunities for children to check by themselves for reflecting on what should be improved.

Following figure 7 is the result of the self-evaluations by teachers on the lesson planning checklist in order to verify their instruction method and the problem solving approach have been appropriate or not. Figure 7 compares the achievement at start time with that of 1.5 years later.

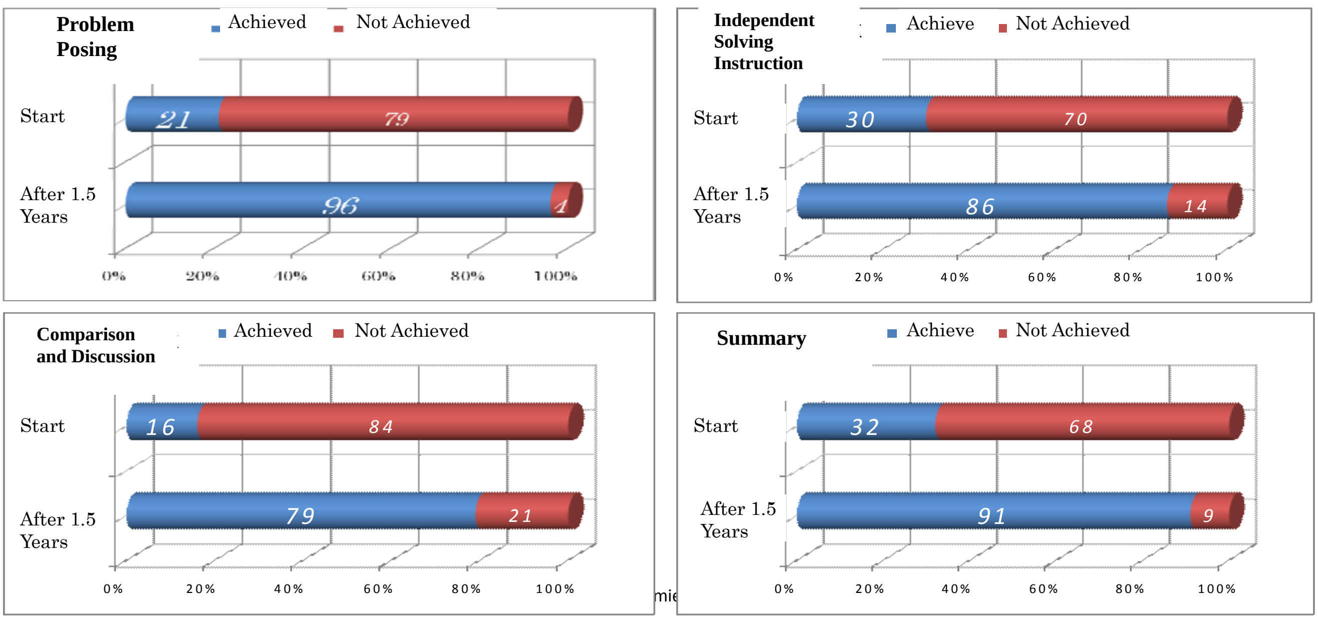


Figure 7. Improvements in Teacher Instruction as Measured with the Lesson Planning Checklist

At the beginning of this research (1.5 years before the lesson study open school), teachers were not sure of the meanings of the words listed on the lesson planning checklist. By taking on the challenge of this project throughout the entire school for one and a half years, the teachers gained confidence in their instruction method. Through the improvement of teachers' teaching practices through the school lesson study project in only one and half years, teachers teaching methods are improved and then, children's achievement are improved beyond mathematics. It was the result of collaborative lesson studies by Ozone Elementary School teachers.

How can a school level lesson study be implemented?

Necessary Conditions for Success: In the case of Japan, Japanese teachers have obligation of self-study and training by themselves (Isoda et al. 2007). Japanese schools have several departments run by teachers and each school has a study department which plans the lesson study topic for whole school level and a training program, and manages lesson study project through the year. A head teacher of the study department and the principal usually collaborate and encourage teachers' lesson study throughout the year. For implementing the lesson study project in mathematics, the head teacher must know mathematics teaching, problem solving approach, and lesson study. At the same time, he/she has to plan the step-by-step progress of teachers teaching abilities: Lesson Planning Checklist is a tool for fostering teachers. In the case of Ozone Elementary School, the head teacher shows their model practice and explains the meaning of the list. In the process of lesson study, the head teacher and the principal participate in the editing process of lesson plan and provide a lot of ideas for teaching. Before actually conducting a lesson, a teacher tries to simulate his/her plan on the blackboard to confirm. In the lesson observation, other teachers observe the class with the same checking lists and at the reflection time after the lesson, they confirm if the lesson was conducted in accordance with the list or not and discuss the necessary preparations for achievement. Through these activities, teachers can share the aims of checklists and be able to give better lessons than in the past, gaining more self-confidence.

For Adaptation into APEC economies: The Problem Solving Approach distinguishes a problem (or a task given by teacher) and a problematic (or a problem posed by children). To differentiate this, it is necessary for comparison and discussion because the Problem Solving Approach is not aimed to solve a problem (or a task) but teaching an objective of lesson through solving a problematic (or a problem). This approach is not easy for a novice teacher because he/she usually tries to teach how to solve a given problem, faces various unexpected answers including misunderstanding and focuses on teaching objective in the process of discussion.

On the other hand, the open-ended approach which is using open-ended problem is easier approach because it aims to solve problems. There are no inappropriate solutions because conditions of a problem are not enough to get necessary solutions. Thus, every child enjoys other's presentation because they just enjoy difference of reasoning and not necessary to learn new content. The open-ended approach is easier for letting teachers know new ways of teaching approach such as problem-solving, posing problem, independent solving, comparison and discussion, and summary and application. This is because if a teacher gives a well-known open-ended problem, most children may produce expected answers and the teacher can ask them to present their answers more easily. With this approach, every child is able to present his/her way of thinking. So if there is no model teacher for the Problem Solving Approach, the open-ended approach is preferable approach to introduce.

In the case of Japan, the open-ended approach has been practiced since even before World War II but it was in 1970s that was recognized on it name. In the case of APEC lesson study projects, for instance, Thailand has introduced the open-ended approach instead of the Problem Solving Approach because it was done through practicum by pre-serves teachers and the open-ended approach is a very good approach to change traditional teachers' teaching belief and children's mind setting (Inprasitha 2006).

<u>For sharing the methodology</u>: The checklists are already introduced in Chile in Spanish (Isoda and Olfos 2009). In Thailand, the approach was introduced through a week lesson study seminar by Isoda with Inprasitha in 2009. The Thailand version is under the process of publication (Isoda and Inprasitha, under preparation). English translation is available only partially (Isoda, under preparation).

References

Inprasihta, M. (2006). Open-Ended Approach and Teacher Education, Tsukuba Journal of Educational Study in Mathematics, Japan: University of Tsukuba. vol.25 169-177.

Isoda, M., Stephens, M., Ohara, Y. Miyakawa, T. edited (2007). Japanese Lesson Study in Mathematics, Singapore: World Scientific.

Isoda, M., Olfos, R., (2009). El Enfoque de Resolución de Problemas : En la Enseñanza de la Matemática, Valparaíso : Ediciones Universitarias de Valparaíso. (Written in Spanish).

Isoda, M. et al. (2009). Designing Problem Solving Class with Basic Standards given by check sheets, Japan: Meijitosyo-publisher (written in Japanese).

Stigler, J., Hiebert, J. (1999). The Teaching Gap. New York : Free Press.

APEC Lesson Study Project Websites (Since 2006-):
At the CRICED, University of Tsukuba, Japan: http://www.criced.tsukuba.ac.jp/math/apec/
At the CRME, Khon Kaen University, Thailand:
http://www.crme.kku.ac.th/Home_APEC.htm

JICA Website in relation to Japanese Experience on Education (Multi Language):
http://www.jica.or.id/english/publications/reports/study/topical/educational/index.html

High School Competency Exams in Hong Kong, China and Teaching Training Programme

CHENG Chun Chor Litwin, The Hong Kong Institute of Education

February 2010

Introduction

The present mathematics curriculum in Hong Kong, China is as follow:

Students will need to do 6 years of primary education and 5 years of secondary education. After the secondary education, students will do a 2 years Advanced –Level education. The following table describe the present curriculum structure in Hong Kong, China. However, a new secondary curriculum take effect from 2009, and by the year 2012, all students in Hong Kong, China will do a 6 year secondary programme (3 year junior secondary and 3 year senior secondary).

Duration	Subjects	
2 years	(A-Level , 2 years) Pure Mathematics Applied mathematics	AS Level, 1 years Applied Mathematics Mathematics and Statistics
5 years	Secondary Mathematics (taken by all students) Additional Mathematics (usually taken by students in science stream)	
6 years	Primary mathematics	

A pass grade in Secondary Mathematics is a requirement for students going to A-Level studies. Hence all Hong Kong, China students need to take this subject. And the discussion of the achievement in this subject may serve the requirement in discussing the standard of High School Graduation/Competency Exams in Hong Kong, China.

For Secondary Mathematics, around 60,000 to 68,000 students each year (day school first attempt) take this paper. From 1999 to 2009, the passing rate of Secondary Mathematics is around 71% to 76%. However, there is no information on the pass mark of getting a pass grade.

For Additional Mathematics, the number of students (day school first attempt) taking this papers is around 17500 to 18500. From 1999 to 2009, the passing rate of Additional Mathematics is around 81% to 85%.

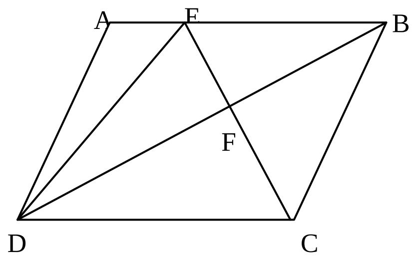
The Secondary Mathematics paper consisted two parts, part one is the multiple choices and part 2 is conventional questions. In part 1, there are 54 multiple choice questions. It is further divided into two sections. Section A consist of 36 questions, which are more fundamental and section B consist of 18 questions which is more difficult. Usually, more than 70% of the students can correctly answer 18 of the 54 questions. Other more difficult questions serve to discriminate the standards of the students.

The following are some examples of attempt by students in Secondary Mathematics.

For example, in one year, 60 % of the students could not see or use the property “two triangles has the same base and same height have the same area”

Question for concept:
In the figure, ABCD is a parallelogram;
E is a point on Line AB.
If EC and BD intersect at F,
find the ratio of the area

of ⊿DEF and ⊿CBF

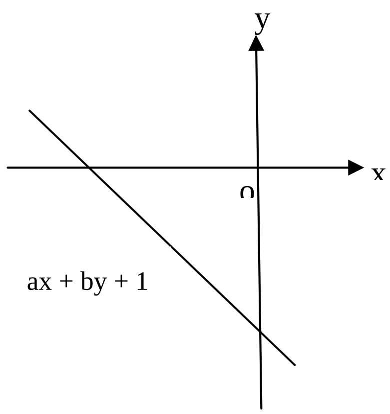


Of the 4 options, only 40% of the students chose the correct option “1:1”. Many students did not see that⊿CDE and⊿BCD has the same base and same height.

Question for concept and procedure:
The figure describe the straight line

$ax + by + 1 = 0$。

.which of the following are correct？



The correct option (A) “a > 0 and b > 0” has a response percentage of 35%. However, the incorrect option (D) $a < 0$ and $b < 0$ has a very high response percentage of 33%.

Example 3

Students are not familiar with transformation of structure. For example, when Reminder Theorem is examined in a way that involves transformation, only 19% of them could answer correctly.

Question for mathematical thinking: If f(x) is a polynomial, and f(x) is divisible by (x-1), then which of the following is a factor of f(2x+1).

The correct option with answer "x" has only a response of 19%. The highest response percentage of a wrong option "2x-1" is 35%. Other incorrect option like "2x+1" attract 27% of the reply.

The program for promoting good Mathematics teaching in Hong Kong, China

In order to promote good practice in mathematics teaching, the Education Bureau of Hong Kong has been organizing a lot of in-service training programme for teachers. Some of them are offered by University and some are by the Bureau staff.

One of the project is lesson study, a project holds in Hong Kong Institute of Education. Primary school teachers can work with staff in University to establish a framework for lesson observation and improve their mathematics teaching. The programme has lasted for 5 years and responses from school are positive.

Another project at the Hong Kong Institute of Education is the study of assessment system, helping school to monitoring their system of assessment in subject including mathematics. This helps school teachers to establish assessment items in mathematics.

A third system project in Hong Kong, China is the invitation of teachers from Mainland China to come to Hong Kong, China and teach for half year. Their planning of teaching is discussed and their teaching can be observed. These teacher also serve as "resources teachers" to help the local school teachers to establish an effective teaching plan.

A fourth system programme organized by the Education Bureau is to assigned staff to support local school to develop school based activities. Each year, about 20 schools in Hong Kong, China are involved in the project. The support staff will work with school teachers to form some school based teaching activities and also hold discussion to ensure teaching a topic at different level effectively.

A fifth system programme is a 5-weeks teacher training programme, teachers come to the Institute to study technique of mathematics teaching and developing materials doing in-depth discussion on teaching selected topics, and also a brief knowledge of most recent research results and information related to high school mathematics teaching. A teacher should be able to develop handful of related materials.

The 5 weeks programme

For a long time, teachers are trained to teach according to what is in the mathematics textbook, and students are expected to answer question within the textbook content. For those mathematics questions that did not appear in the textbook, they are usually ignored.

The 5-weeks programme enjoy a lot of positive responses from teachers, as the course not only provide theoretical framework, but also practical technique of teaching and development of teaching material. The in-service teachers attended the programme and learn the latest technique or skills to teach certain topics in mathematics. If a program can satisfy the needs of teacher to allow them to develop their own material, it enjoy more positive support, especially that the topics that teachers think difficult to teach are taken for discussion and teachers benefit from it. They may use new approaches to teach a certain topic, or they learn to use various different approaches to teach the same topic.

However, the technique wise training is not always effective in teacher education. It is more important to change the mindset of the teachers so that not only they accept the technique and they are ready to use such technique, but also that they can reproduce some of the technique themselves. If they could not produce teaching technique of their own, then we will need teacher retraining again and again, making such retraining programme not very effective.

The present situation of mathematics teaching - generalist and specialists

A mathematics teacher should be able to synthesis knowledge of mathematics, developmental and psychological theories, and the pedagogical knowledge for teaching the subjects. So, a teacher needs training in both mathematics knowledge and curriculum theories in teaching mathematics.

In Hong Kong, China, most of the mathematics teachers are math trained. But for teachers at the lower forms, many are graduates with an engineering degree or even not trained in mathematics. What they need is the specific technique in teaching the topic, and most of them are included in the textbook.

Knowing the technique of teaching mathematics is not enough for good mathematics education. Teachers should be able to do exploration in mathematics if they wish to enhance their teaching. By doing exploration, teachers have the ability to know the structure, allow teachers to construct problems of mathematics.

There are in general two kinds of mathematics teachers, the specialist and the generalist. They can be divided into three type of mathematics teaching, transmission type, discovery type and connectionist type. Transmission type is those who worked with a standard procedure in calculation, discovery type is those who wish to emphasis on procedure that are practical and can be discovered by students. Connectionist is those who try to connect what is learned by students and the content of mathematics. By exploring the content of some mathematics investigation, teachers understand the process of their own thinking. Connectionist types are more likely to be highly effective teachers.

	Transmission approach	Discovery approach	Connection approach
Generalist training	Very Likely	Less Likely	Very unlikely
Specialist training	Very Likely	Likely	Likely

Those teachers who are generalist are usually transmission type than discovery type. This is because they may not be able to understand the structure of mathematics and the structure of the problem and solution. And direct teaching is a safe channel.

The other extreme end of the generalist approach is what we now call “facilitating learning”, in which teacher is a facilitator and did not teach. The results of such teaching may be disastrous, as the term “facilitator” helps those generalists to hide from doing real knowledge in mathematics or not even prepare their teaching.

For that teacher with specialist training in mathematics, they can choose to use transmission approach. However, they are more likely to use discovery approach as they could use their mathematics knowledge to guide students to do discovery work.

Examples that specialist training could use connection approach

The first step is for teachers to get good and interesting mathematics questions. And from then they can obtain some basic technique in solving the mathematics problem. And the last step is for them to observe and obtain the structure of the questions, and relate the structure to the solution. For example, the teaching of the following mathematics structure.

Question：

Express the fraction $\frac{n-1}{n}$ as a sum of 3 fractions.

$\frac{n-1}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

Why the question? Teacher can easy find connected mathematics problem.

Question：

An old man has 17 horses and he is going to leave $\frac{1}{2}$ of his horses to his elder son, $\frac{1}{3}$ of his horses to his second son, $\frac{1}{9}$ of his horses to his youngest son.

The answer is possible when one horse is borrowed to make up to 18. And it is because of the following mathematical structure, $\frac{1}{2} + \frac{1}{3} + \frac{1}{9} = \frac{17}{18}$, which has the same structure as $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{n-1}{n}$.

To show the general solution, we know that as $\frac{n-1}{n} < 1$, a ≥ 2. And a can only be 2.

As the sum of the 3 fractions is less than 1, one of them should be greater than $\frac{1}{3}$, and hence $\frac{1}{2}$ is one of the three fractions.

Take $\frac{1}{2}$ as one of the 3 fraction, then we have $\frac{n-1}{n} = \frac{1}{2} + \frac{1}{b} + \frac{1}{c}$.

To maximize the sum of the 3 fractions, we have $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} = \frac{47}{60}$.

Using trail and error, $\frac{n-1}{n} = \frac{1}{2} + \frac{1}{3} + \frac{1}{c}$ and obtain $\frac{1}{c} = \frac{1}{6} - \frac{1}{n}$。

As n is greater than c, so $\frac{1}{c} = \frac{1}{6} - \frac{1}{c}$. That is, c ≤ 12. As $\frac{1}{c}$ is less than $\frac{1}{6}$. So we have c ≥ 7.

We have the following 7 answer for $\frac{n-1}{n} = \frac{1}{2} + \frac{1}{b} + \frac{1}{c}$,

$\frac{7}{8} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, $\frac{11}{12} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6}$, $\frac{11}{12} = \frac{1}{2} + \frac{1}{3} + \frac{1}{12}$.

$\frac{17}{18} = \frac{1}{2} + \frac{1}{3} + \frac{1}{9}$, $\frac{19}{20} = \frac{1}{2} + \frac{1}{4} + \frac{1}{5}$, $\frac{23}{24} = \frac{1}{2} + \frac{1}{3} + \frac{1}{8}$.

$\frac{41}{42} = \frac{1}{2} + \frac{1}{3} + \frac{1}{7}$.

Further Exploration 1：

Write $\frac{n+1}{n}$ as sum of three unit- fractions

$\frac{n+1}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

Further Exploration 2：

Write $\frac{n-1}{n}$ as sum of four unit-fractions

$\frac{n-1}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$。

Example in applying technique of solving problem

For teachers to be able to connect his lessons, he is able to found example which extend the mathematics structure. For example, the technique for solving the following question is basic in high school, but extending the application needs more different type of questions

First, observe the process of solution of the following problem

$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}+\frac{1}{56}+\frac{1}{72}$

$= \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\frac{1}{5\times 6}+\frac{1}{6\times 7}+\frac{1}{7\times 8}+\frac{1}{8\times 9}$

$= (1-\frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3}-\frac{1}{4}) + (\frac{1}{4}-\frac{1}{5}) + \ldots.. + (\frac{1}{8}-\frac{1}{9})$

$= \frac{8}{9}$.

> Exploration 1：
>
> By using the above process, find a structure and solve the following question:
>
> $$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\frac{1}{7\times9}+\frac{1}{9\times11}+\frac{1}{11\times13}+\frac{1}{13\times15}+\frac{1}{15\times17}$$

The idea of splitting the terms and then cancelled can be extended to other examples.

> Exploration 2：
>
> Simplify $\frac{1}{\sin 2\alpha}$ + $\frac{1}{\sin 4\alpha}$ + ….+ $\frac{1}{\sin 2^n\alpha}$。

As $2\cos^2\theta - \cos(2\theta) = 1$, hence $\frac{1}{\sin 2\alpha}$ = $\frac{2\cos^2\alpha - \cos 2\alpha}{\sin 2\alpha}$ = $\frac{2\cos^2\alpha}{\sin 2\alpha}$ - $\frac{\cos 2\alpha}{\sin 2\alpha}$ = $\frac{2\cos^2\alpha}{2\sin\alpha\cos\alpha}$ - $\frac{\cos 2\alpha}{\sin 2\alpha}$

$= \cot\alpha - \cot2\alpha$。

Similarly, $\frac{1}{\sin 4\alpha}$ = $\cot2\alpha - \cot4\alpha$,

$\frac{1}{\sin 2^n\alpha}$ $= \cot2^{n-1}\alpha - \cot^{2n}\alpha$。

Hence $\frac{1}{\sin 2\alpha}$ + $\frac{1}{\sin 4\alpha}$ + ….+ $\frac{1}{\sin 2^n\alpha}$

$= \cot\alpha - \cot2\alpha + \cot2\alpha - \cot4\alpha + \ldots + \cot2^{n-1}\alpha - \cot^{2n}\alpha$

$= \cot\alpha - \cot^{2n}\alpha$。

Reverse process, combining terms

Exploration 3：

Find the value of sin10° sin30° sin50° sin70°.

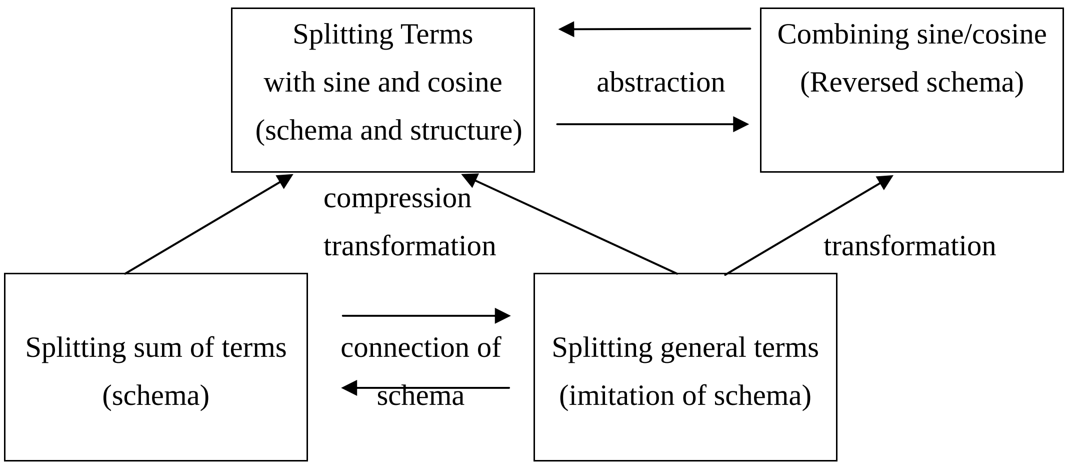
By the relation of sine and cosine, try using cos10° cos30° cos50° cos70°
And make use of the formula sinAcosA.
Let A = (sin10° sin30° sin50° sin70°), B = (cos10° cos30° cos50° cos70°)

then AB = $\frac{1}{16}$(sin20° sin60° sin100° sin140°)

= $\frac{1}{16}$(cos10° cos30° cos50° cos70°) = $\frac{1}{16}$B。

As B ≠ 0, A = $\frac{1}{16}$。

The process of exploration in the examples of using technique of splitting terms can be described by the following framework.



An example used in the 5-week programme in mathematics teacher training

The following is some examples in the programme, Finding the area of triangle and quadrilateral.

Teachers will analyze with their class the property of the area of a triangle.

If $S_{(a,b,c)}$ denote the area of a triangle with sides a, b, and c, then there are properties that need to go through with students.

We will have $S_{(a,b,c)} = S_{(b,c,a)} = S_{(c,a,b)}$. This is the symmetric property.

Also, as the triangle is not biased with any three sides, the formula of the area of the triangle should include symmetrical expressions such as $a+b+c$ (degree one) , $a^2+b^2+c^2$, $ab+bc+ca$ $(a+b+c)^2$ (degree two), (a+b+c)(ab+bc+ca) (degree three) etc.

Also, the dimension of the area of a triangle must be the square of length.

Discussion with the class come to the summary that, the formula of area of a triangle could look like this:

$S_{(a,b,c)} = K\sqrt{(a+b+c)(a^3+b^3+c^3)}$, as it is symmetrical and satisfy the dimensional requirement.

However, another important property of a triangle is the "sum of any two sides is longer than the third side", so there should be a factor in the form of b + c – a > 0. And b + c – a = 0 is a boundary condition. If a + b = c or b + c = a or c + a = b, then the area of the triangle will be zero. Hence $S_{(a,b,c)}$ consisted the three factors (a + b –c), (b + c – a), and (c + a – b).

Perhaps the formula is $S_{(a,b,c)} = K[(a+b-c)(b+c-a)(c+a-b)]^{\frac{2}{3}}$, but the formula consists of only three factors and it will give negative values of a area, such as a = 1, b = 2, and c = 100. and such triangle does not exist.

Hence the guess is that another factor $(a+b+c)$ exist, and the formula is in the form $S_{(a,b,c)} = K(a+b+c)(b+c-a)(c+a-b)$. According to the requirement of dimension, it is rewritten as $S_{(a,b,c)} = K\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}$ 。

Here, the conjectured formula satisfies our requirement in symmetry, dimensional and boundary condition. And we test to find out the numerical values of K. By putting in the information of a right angle triangle, we have $c^2 = a^2 + b^2$, and the area $\frac{1}{2}ab$.

Then

$$K\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}$$

$$= K\sqrt{[(a+b+c)(a+b-c)][c+(b-a)][c-(b-a)]}$$

$$= K\sqrt{\left[(a+b)^2-c^2\right]\left[c^2-(b-a)^2\right]}$$

$$= K\sqrt{\left(a^2+b^2+2ab-c^2\right)\left(c^2-b^2+2ab-a^2\right)}$$

$$= K\sqrt{2ab \times 2ab}$$

$$= 2Kab$$

By $\frac{1}{2}ab$ = 2Kab, we have K=$\frac{1}{4}$. Hence the formula is deduced to be

$$S_{(a,b,c)} = \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}$$

The above deduction is based on assumption and conditions, and need to be proved.

If we checked the validity of the formula again, using the data of a equilateral triangle with side a, and area $\frac{\sqrt{3}}{4}a^2$, the area given by the formula is $\frac{1}{4}\sqrt{3a \times a \times a \times a}^2 = \frac{\sqrt{3}}{4}a$, which in line with fact.

However, a mathematical proof is more than verification with examples. The following is the proof of the formula, though the proof itself is not the same as it was derived 2000 year ago.

Using area = $\frac{1}{2}$ ab sinC to device the formula of area of a triangle with sides a, b and c.

We start with a triangle ABC and $S_{(a,b,c)} = \frac{1}{2}$ ab sinC.

Squaring, $S^2_{(a,b,c)} = \frac{1}{4}a^2b^2\sin^2 C = \frac{1}{4}a^2b^2(1-\cos^2 C)$。

By Cosine Rule, $\cos C = \frac{a^2+b^2-c^2}{2ab}$, after substitution,

$$S^2_{(a,b,c)} = \frac{1}{4}a^2b^2\,\frac{4a^2b^2-\left(a^2+b^2-c^2\right)^2}{4a^2b^2}$$

$$= \frac{1}{16}\left[\left(2ab + a^2 + b^2 - c^2\right)\left(2ab - a^2 - b^2 + c^2\right)\right]$$

$$= \frac{1}{16}\left[(a+b)^2 - c^2\right]\left[c^2 - (a-b)^2\right]$$

$$= \frac{1}{16}(a+b+c)(a+b-c)(c+a-b)(c+b-a)$$

If we put $p = \frac{1}{2}(a+b+c)$, then $\frac{1}{2}(a+b-c)$ = p – c , $\frac{1}{2}(b+c-a)$ = p – a , $\frac{1}{2}(c+a-b)$ = p – b。

$\Rightarrow S^2_{(a,b,c)}$ = p(p-a)(p-b)(p-c)

$\Rightarrow S_{(a,b,c)} = \sqrt{p(p-a)(p-b)(p-c)}$

The formula $S_{(a,b,c)}$ now satisfy the requirement that：

(i) Area are positive, $S_{(a,b,c)} \geqq 0$。

(ii) Formula in area of triangle is symmetric, $S_{(a,b,c)} = S_{(a,b,c)} = S_{(c,a,b)}$

(iii) Dimension of area $= L^2$

(iv) Boundary conditions, when $a+b=c$, $b+c=a$, $c+a=b$, $S_{(a,b,c)} = 0$

(v) Proportion of area, $S_{(\lambda a,\lambda b,\lambda c)} = \lambda^2 S_{(a,b,c)}$, $\lambda \geqq 0$。

The discussion continues to explore the relation of the Heron's Formula and the Pythagoras theorem.

Exploration

If right angle triangle with two adjacent sides a, b, show that we can obtain the relation $c^2 = a^2 + b^2$.

By using the formula of area and also it equals to $\frac{ab}{2}$, that is

$\frac{1}{16}(a+b+c)(a+b-c)(c+a-b)(c+b-a) = (\frac{ab}{2})^2$, students are required to deduce the result $c^2 = a^2 + b^2$.

In China, around 1000 years ago, a scholar "Zhun Kiu Siu" deduce a formula for the area of a triangle in his book "Nine Chapters in Mathematics" The formula is

$$S_{(a,b,c)} = \frac{1}{2}\sqrt{b^2a^2 - \left(\frac{b^2+a^2-c^2}{2}\right)^2}$$

Exploration

Deduce the Heron's formula from the Zhun's formula.

From $S^2_{(a,b,c)} = \frac{1}{4}\left[b^2a^2 - \left(\frac{b^2+a^2-c^2}{2}\right)^2\right]$

$$= \frac{1}{16}\left[(2ab)^2 - (b^2+a^2-c^2)^2\right]$$

$$= \frac{1}{16}\left[(b+a)^2 - c^2\right]\left[c^2 - (b-a)^2\right]$$

$$= \frac{1}{16}(a+b+c)(a+c-b)(b+c-a)(b+a-c)$$

$$= p(p-a)(p-b)(p-c)$$

There are many possible ways to deduce the formula of area of triangle by using nowadays technique.

Exploration

Deduce the formula for area of triangle from $\sin A = \dfrac{2\Delta ABC}{bc}$, $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$ and $\sin^2 A + \cos^2 A = 1$.

We substitute $\sin A = \dfrac{2\Delta ABC}{bc}$ and $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$ into $\sin^2 A + \cos^2 A = 1$, then

$$\frac{4(\Delta ABC)^2}{b^2c^2} + \frac{(b^2+c^2-a^2)^2}{4b^2c^2} = 1。$$

$$(\Delta ABC)^2 = \frac{1}{16}(4b^2c^2 - (b^2+c^2-a^2)^2)$$

$$= \frac{-1}{16}(b^2+c^2-a^2-2bc)(b^2+c^2-a^2+2bc)$$

$$= \frac{1}{16}[(b+c)^2-a^2][a^2-(b-c)^2]$$

Another approach can be used as follow:

$$\Delta^2 = \frac{1}{4}b^2c^2\sin^2 A$$

$$= \frac{1}{4}b^2c^2(1-\cos^2 A)$$

$$= \frac{1}{4}b^2c^2(1-\cos A)(1+\cos A)$$

By $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$, we have

$$1 + \cos A = \frac{b^2+c^2-a^2+2bc}{2bc} = \frac{(a+b+c)(b+c-a)}{2bc}$$

$$1 - \cos A = \frac{b^2+c^2-a^2-2bc}{2bc} = \frac{(a-b+c)(a+b-c)}{2bc}$$

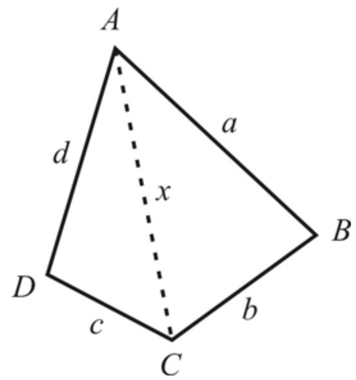
Hence $\Delta^2 = \dfrac{1}{16}(a+b+c)(b+c-a)(c+a-b)(a+b-c)$

Exploration

ABCD is a quadrilateral, AB = a, BC = b, CD = c, DA = d。

Show that the area can be

$$S_{ABCD} = \frac{1}{4}\left(a^2b^2 + c^2d^2 - \left(\frac{(a^2+b^2)-(c^2+d^2)}{2}\right)^2\right) - \frac{1}{2}abcd\cos(B+D)$$



The process of discussion:

AC = x, using cosine rule,

$$a^2 + b^2 - 2ab\cos B = x^2 = c^2 + d^2 - 2cd\cos D$$

$$\Rightarrow \ ab\cos B - cd\cos D = \left(\frac{(a^2+b^2)-(c^2+d^2)}{2}\right)$$

$$\Rightarrow \ (ab\cos B - cd\cos D)^2 = \left(\frac{(a^2+b^2)-(c^2+d^2)}{2}\right)^2$$

$$\Rightarrow \ (ab\cos B)^2 + (cd\cos D)^2 = \left(\frac{(a^2+b^2)-(c^2+d^2)}{2}\right)^2 + 2abcd\cos B\cos D$$

$$S^2_{ABCD} = \left(\frac{1}{2}ab\sin B + \frac{1}{2}cd\sin D\right)^2$$

$$= \frac{1}{4}\left((ab\sin B)^2 + (cd\sin D)^2 - 2abcd\sin B\sin D\right)$$

$$= \frac{1}{4}\left((ab)^2(1-\cos^2 B) + (cd)^2(1-\cos^2 D) + 2abcd\sin B\sin D\right)$$

$$= \frac{1}{4}\left((ab)^2 + (cd)^2 - \left(\left(\frac{(a^2+b^2)-(c^2+d^2)}{2}\right)^2 + 2abcd\cos B\cos D\right) + 2abcd\sin B\sin D\right)$$

$$= \frac{1}{4}\left(a^2b^2 + c^2d^2 - \left(\left(\frac{(a^2+b^2)-(c^2+d^2)}{2}\right)^2\right) - \frac{1}{2}abcd(\cos B\cos D - \sin B\sin D\right)$$

$$= \frac{1}{4}\left(a^2b^2 + c^2d^2 - \left(\frac{(a^2+b^2)-(c^2+d^2)}{2}\right)^2\right) - \frac{1}{2}abcd\cos(B+D)$$

B and D are two angle of the quadrilateral. The result can apply to convex and concave quadrilateral.

For the above area to achieve maximum value, cos(B+D) = -1, That is B + D = 180°, which means ABCD is a quadrilateral inscribed in a circle.

Hence for a inscribed quadrilateral,

$$S^2_{ABCD} = \frac{1}{4}\left(a^2b^2 + c^2d^2 - \left(\frac{(a^2+b^2)-(c^2+d^2)}{2}\right)^2\right) + \frac{1}{2}abcd$$

Exploration:

We know that a triangle with sides a, b, c has area =

$\sqrt{p(p-a)(p-b)(p-c)}$, where p is half the perimeter of the triangle.

Now for an inscribed quadrilateral ABCD, AB = a, BC = b, CD = c, DA = d,

would it be possible that the area = $\sqrt{(p-a)(p-b)(p-c)(p-d)}$, where p is half the perimeter of the quadrilateral.

That is , will

$$S_{ABCD} = \frac{1}{4}\left(a^2b^2 + c^2d^2 - \left(\frac{(a^2+b^2)-(c^2+d^2)}{2}\right)^2\right) + \frac{1}{2}abcd$$

$= \sqrt{(p-a)(p-b)(p-c)(p-d)}$?

Mathematical thinking will first check whether the formula of the area of the quadrilateral $\sqrt{(p-a)(p-b)(p-c)(p-d)}$ can hold at a special case d = 0, which is a triangle.

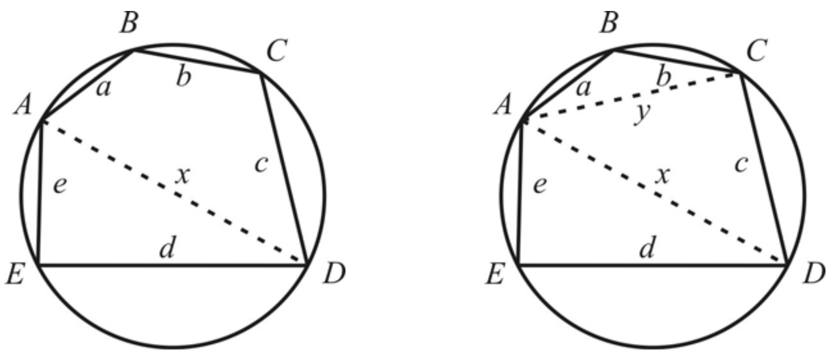
As $S_{(a,b,c,0)} = \sqrt{(p-a)(p-b)(p-c)(p-0)} = \sqrt{p(p-a)(p-b)(p-c)}$ 。

Hence the formula holds for the special case. The next step is to deduce that the formula is correct. This will be an exploration for the students.

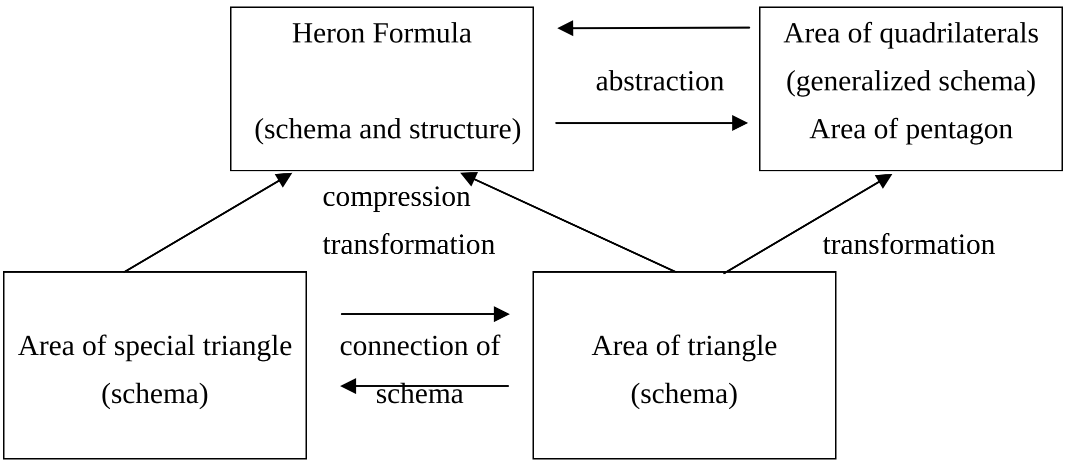
The exploration can carry on to the case of area of a pentagon.

Exploration

Can we find the area of a pentagon through dissecting the figure into a quadrilateral and a triangle.



The process of exploration in the examples of area of triangle and quadrilateral



References

Hong Kong Examination and Assessment Authority (2008), Examiner Report. Hong Kong SAR.

Education Department (1999), Secondary Mathematics Curriculum, Hong Kong SAR.

Formative Assessment: a key component in improving student achievement in mathematics in New Zealand

Ian Stevens, Ministry of Education

February 2010

Introduction

> "To be numerate is to have the ability and inclination to use mathematics effectively – at home, at work and in the community.[1]"

The goal of mathematics education in New Zealand is for all students to become numerate, leaving school with a positive attitude towards mathematics, coupled with an understanding and ability to use mathematics effectively whenever needed.

The New Zealand national curriculum[2] is the official policy relating to teaching and learning in schools. It is a statement of what New Zealand deems important in education. Its principal function is to set the direction for student learning and to provide guidance for schools as they design and review their curriculum. It takes as its starting point a vision of young people as lifelong learners who are confident and creative, connected, and actively involved. It sets out values that are to be encouraged, modeled, and explored, defines five key competencies that are critical to sustained learning and effective participation in society and describes the outcomes for students in eight interconnecting learning areas.

Mathematics is a key learning area or subject in the New Zealand curriculum and essential to students becoming confident and creative, connected and active learners. The learning area in New Zealand is called mathematics and statistics. Mathematics is the exploration and use of patterns and relationships in quantities, space, and time and statistics is the exploration and use of patterns and relationships in data. These two disciplines are related but are different ways of thinking about and solving problems. Both equip students with effective means for investigating, interpreting, explaining, and making sense of the world in which they live.

The Third International Mathematics and Science Study (TIMSS) in 1994/1995 identified the achievement of New Zealand students as being significantly below the international mean in mathematics and science[3]. In response to these results the government established the Mathematics and Science Taskforce to provide advice on how to improve the teaching of mathematics and science in New Zealand schools. The taskforce highlighted a number of overriding priorities in relation to raising performance in mathematics, in particular the need to raise teachers and parents expectations of success, improve the professional skills, knowledge and confidence of teachers, provide resources and professional development for teachers to

[1] The definition and goal of numeracy, New Zealand Ministry of Education, 2002
[2] *The New Zealand Curriculum*, 2007 for English medium schools and *Te Marautanga o Aotearoa*, 2008 for Māori medium schools, Ministry of Education. The terms English medium and Māori medium are used to indicate the language of instruction.
[3] New Zealand's TIMSS results can be found at http://www.educationcounts.govt.nz/publications/series/2571

support them in implementing the curriculum, and lift Māori and Pacific Island students' levels of achievement. These strategic priorities led to the design and implementation of the Numeracy Development Project[4]. The project was developed for the two languages of instruction used in New Zealand schooling English medium, starting in 2000, and Te Poutama Tau, Māori medium, starting in 2002. Each of the projects informs the ongoing development of the other.

A system wide focus on improvement was adopted, rather than a focus on specific groups of students or regions. This was in response to the diversity of New Zealand students and schools. "New Zealand has a wide spread of achievement compared to other highly performing economies – with relatively large proportions at both a very high level and also at a very low level."[5] Students from all ethnicities, socio economic backgrounds and genders are represented in both the highest and lowest performing groups in New Zealand. However, Māori students, Pacific Island students and students from low socio economic backgrounds are proportionally over represented in the lowest performing group. The number of small schools in New Zealand, many rural and isolated, is another feature that needed to be considered in planning for improvement.[6]

Numeracy Development Project

The Numeracy Development Project and Te Poutama Tau, major government funded national strategies, aim to improve student achievement through improving the professional capability of teachers.

The strategic objectives of the Numeracy Development Project and Te Poutama Tau are:

- Improved student achievement in mathematics,
- Improved knowledge, skills and confidence of teachers in mathematics,
- Improved achievement of Māori and Pacific Island students, and
- Māori language revitalization (Te Poutama Tau).

A dynamic and evolutionary approach to the design and implementation of the project is a key feature, with assessment, research and evaluation used to inform the ongoing development at the classroom, school and system level. The design drew on evidence from mathematics education, effective teaching, teacher learning, effective professional development, educational change and system reform as well as from the on-going research and evaluation associated with the project. Since the beginning of the project there have been 103 research and evaluation papers published by the Ministry of Education[7].

The premise of the project to improve student achievement by improving the professional capability of teachers is based on the belief that teachers are key figures in changing the way in which mathematics is taught and learned in schools. Their subject matter, pedagogical knowledge and assessment capability are critical factors in the teaching and learning of mathematics. The effective teacher has a thorough and deep understanding of the subject matter

[4] The terms numeracy and mathematics are used interchangeably in New Zealand with specific information located in the Mathematics and Statistics learning area of *The New Zealand Curriculum*.
[5] Satherley, (2010)
[6] Almost 50% of New Zealand's primary schools have less than 150 students and almost 20% of schools have only two teachers. 5 to 13 years old students attend New Zealand primary schools.
[7] Papers can be downloaded from www.nzmaths.co.nz/annual-evaluation-reports-and-compendium-papers

to be taught, how students are likely to learn it, the difficulties and misunderstandings they are likely to encounter, and effective formative assessment practices.

The location of the professional development was also a key feature in the design of the project. "Professional learning is strongly shaped by the context in which the teacher practises. This is usually the classroom, which, in turn, is strongly influenced by the wider school culture and the community and society in which the school is situated. Teachers' daily experiences in their practice context shape their understandings, and their understandings shape their experiences."[8] School advisers, external to the school, support teachers and school leaders by leading workshops, visiting teacher's classrooms to model ideas with students, observing and giving feedback. They also providing resources, assist in the analysis of student achievement information, and support the learning needs of the school's teachers and leaders as needed.

The professional development model for the Numeracy Development Project and Te Poutama Tau is based around teachers understanding and using three key pedagogical tools;

- The Number Framework,
- Diagnostic Interview, and
- Strategy Teaching Model.

These three tools together enable teachers to developing their knowledge, ability and confidence in knowing their students learning needs, and be able to provide a quality teaching and learning programme. The number framework provides a structure for teaching and learning, the diagnostic interview finds out where on the framework students are, as well as the next learning steps, and the strategy teaching model guides how to teach this next step.

The project starts with teachers being introduced to the number framework through workshops, which includes videos of students articulating their thinking. Teachers conduct the diagnostic interview with each student in their class, initially with support from the adviser. The resulting student achievement information is used to develop a teaching programme based on the learning needs of the students. Through a series of workshops and classroom visits by the adviser, teachers gradually improve their professional capability. Their teaching becomes based on the learning needs of their students rather than on a predetermined programme based on the age of the students or level of schooling.

The Number Framework

The structure of the framework is based on the idea that there are increasingly sophisticated ways of thinking mathematically and that it is useful to set out the different types of thinking as a progression for pedagogical purposes.

The framework is divided into two main sections, strategy and knowledge, each with eight stages of development. The strategy section describes the mental processes students use to solve problems involving numbers and estimate answers. The knowledge section describes the key items of knowledge that students need to learn and be able to quickly recall in order to be able to estimate and solve problems. It is important that students make progress in both sections of the framework. Strong knowledge is essential for students to broaden their strategies across a full

[8] Timperley, 2008

range of numbers, and knowledge is often an essential prerequisite for the development of strategies. The strategy section is based on two broad areas of development, the first based on counting and the second on the notion of part-whole thinking, (Cobb & Wheatly, 1998).

The framework is an important pedagogical tool for teachers as it enables them to become more focused in their teaching through developing their knowledge of how students learn mathematics. Often at the start of the professional development, teachers indicate their vagueness about what they were teaching in mathematics. Contrasting their previous practice, they commented, "I am much more focused in my teaching objectives", "The project has given my teaching more structure." and "It's about giving simple, understandable, credible, reasonable structures for teachers to use." [9] Laying out professional progressions in some detail in the framework enabled in-depth assessment of students' understanding of mathematical ideas which teachers found very helpful, (Higgins & Parsons, 2009).

The Diagnostic Interview

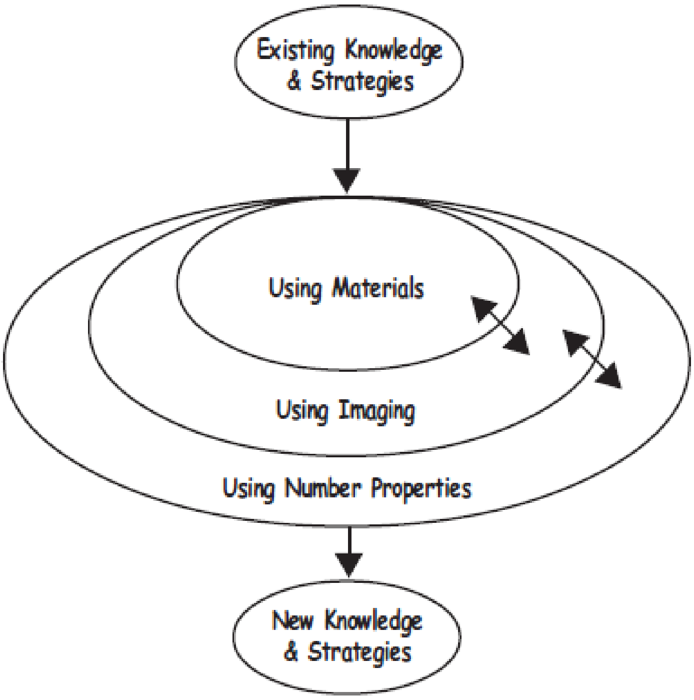
One of the outcomes sought through teachers' participation in the project is increased teacher responsiveness to students' diverse learning needs through seamlessly integrating knowledge of number progressions into their mathematics teaching practice. The diagnostic interview, based on Wright's (1998) work, has been designed to support teachers' development in identifying students' knowledge and strategies and using the evidence as the basis for planning students' next learning sequence. The information gained can also be used to report to parents. The fact that items in the diagnostic interview are aligned to the number framework provides teachers with an enriched knowledge about progressions in learning number. It is "one of the essential triggers" for challenging teacher's beliefs and changing teacher's knowledge and practice, (Higgins & Parsons, 2009).

One of the most powerful outcomes for teachers when first using the diagnostic interview was the overturning of previously held assumptions about the extent of individual student understanding of number concepts. Many teachers commented that through the interview they found that some students whom they had previously thought to have good number understanding struggled in their attempts to explain their answers. Conversely, other students, whom teachers had regarded as having weak knowledge, demonstrated deeper understanding. A teacher expressed how conducting the diagnostic interview with her own students affected her preconceived ideas, (Higgins & Parsons, 2009). Teachers started to think about what students had learnt rather than what they had been taught. Comments like "This group of students won't be able to achieve like the others", "They won't know this because I haven't taught it yet" and "They will know this because I have just taught them" were challenged. Teachers started to shift from focusing on how many answers students got correct to how students worked them out and their level of understanding. The framework and interview gave teachers a clear picture of each student's understanding and ability. This led to the need to develop classroom programmes that were responsive to the students' actual learning needs.

The Strategy Teaching Model

[9] Higgins and Parsons, 2009

The strategy teaching model is interconnected to the framework and the information gained from the diagnostic interview guided teachers in the explicit teaching of strategies (Hughes, 2002). The following diagram outlines the model. The development of any new strategy starts from what the student already knows and can do. The arrows on the diagram illustrate a dynamic relationship between the phases with movement through these phases demonstrating greater degrees of abstraction in a student's thinking. Progression from Using Materials to Using Imaging is usually promoted by the teacher masking materials and asking anticipatory questions about actions on those materials. Progression to Using Number Properties is promoted by increasing the complexity or size of the numbers involved, thus making reliance on the material representation difficult and inefficient. This model was influenced by the P-K theory of Pirie & Kieren (1989).



A feature of the model is the aim of developing students' mental strategies through students explaining their thinking prior to the introduction of the written algorithm. Shifting teachers' practice from the premature introduction of the written algorithm has implications for the organization of classroom learning, the use of presentations, the recording of problem solutions, and parent expectations which need to be considered and planned for.[10]

The framework, interview and model are key parts of a quality teaching and learning programme along with the seven dimensions of quality teaching critical to improving student outcomes identified by research. The dimensions are incorporated throughout the project resources,

[10] Numeracy Development Project, Book 3, Ministry of Education, 2008

discussed at workshops and school visits, and modelled by advisers in classrooms. The seven dimensions outlined by Alton-Lee, 2003[11] are:

- Inclusive classroom climate
- Focused planning
- Problem-centred activities
- Responsive lessons
- Connections
- High expectations
- Equity

The project is supported by a wide range of resources, including the nine Numeracy Development Project books[12] and the Ministry of Education's mathematics curriculum website, www.nzmaths.co.nz. The website contains a wide variety of documents, videos, online professional development opportunities, planning tools, interactive learning tools, lessons and units of work. There is also a small section to support parents in helping their children. The website has both English medium and Māori medium sections.

Implementation Approach

Since 2000, almost all of New Zealand's 2,100 primary schools have been involved in the initial two years of the Numeracy Development Project. This is approximately 29,000 teachers, including approximately 800 Māori medium[13] teachers, and 800,000 students. The average time allocated to each teacher for facilitation and support by an adviser is approximately 13 hours in the first year and 5 hours in the second year, costing approximately $3,300 per teacher. This cost is used to contract and coordinate teams of school advisers to work with teachers and school leaders, to release teachers to conduct the diagnostic interview, to provide resources and equipment, and fund research and evaluations.

The project is centrally coordinated from the national office of the Ministry of Education and regionally led by coordinators with teams of advisers all working together. The regional teams are based at the six main New Zealand universities allowing for synergy between pre-service teacher educators, researchers and school advisers. Access for teachers to university post-graduate mathematics education papers through a fee subsidy is also provided.

Advisers work directly alongside teachers the first time they conducts the diagnostic interview to guide and support their interpretation of the students' responses to items from the interview. The diagnostic interview has three embedded design elements: First, it is designed as a model for the types of questions that teachers might use in teaching students; second, teachers deepen their understanding through the items in diagnostic interview which illustrate the different stages of the number framework; and third, the information gained through the interview enables teachers to develop more specific expectations of student learning. The strategy and knowledge components of the interview build teachers' knowledge of the interconnectedness of mathematical ideas, (Higgins & Parsons, 2009).

[11] Numeracy Development Project, Book 3, Ministry of Education, 2008
[12] Numeracy Development Project books can be downloaded at: http://www.nzmaths.co.nz/numeracy-development-projects-books
[13] The language of instruction is Māori, the language of the indigenous people of New Zealand

Following the diagnostic interview a cycle of workshops and classroom and school visits begins. The workshops follow the order of the project books. Following each workshop advisers visit each teacher's classroom to model the ideas from the workshop. The in-class work of the adviser varies over the course of the project in response to the teachers learning need. The adviser's in-class work includes modelling lessons or parts of lessons, teaching alongside the classroom teacher, and observing the teacher in action followed by feedback and discussion. Supporting planning at both the classroom and school level is also an important role of the adviser along with support the school's mathematics leaders as needed. Initially the adviser takes a lead in the development, alongside the school leaders who are encouraged to be fully involved. As the implementation progresses the school leaders take over leading the development in their school. Near the end of each year the schools leaders, with advisers support if needed, plan ways to sustain the improvements already made and to continue improving teacher capability and student achievement.

To continue supporting schools after being involved in the initial two years of the project, regionally based networks of numeracy lead teachers from each school are organized by advisers. All schools are invited to send one or more lead teachers to these regular meetings where they hear about new resources, new ideas and how others are improving teacher capability and student achievement. This forum also allows lead teachers to support and learn from each other as well as hear from advisers and other experts.

Evaluating the Effectiveness of the Project

A research and evaluation programme investigating the effectiveness of the Numeracy Development Project has played a critical role in the success of the project. The approach adopted to gather evidence has been multi methodological and iterative with a focus on student achievement, the professional practice of teachers and advisers, and sustainability, (Higgins & Parsons, 2009).

Each year researchers are contracted by the Ministry of Education to research and evaluate the project. The research includes studies analysing changes in student achievement and trends in the data over time. As part of the project teachers enter student achievement information in relation to the framework onto a secure website. This information is used to evaluate the project and in planning future professional development. Other studies focus on teacher and adviser practice and the longitudinal effects of the project. This research has both an English medium and a Māori medium setting focus.

The following table is an example of how one Numeracy Development Project school reported the progress in achievement during the first year to their Board of Trustees. The table shows the number of students at each stage of the Number Framework in each year level at three points in the year; February, June, and November.[14]

The blue in the table indicates the level of achievement expected at the end of that year, yellow indicates that they are just below the expected level, grey indicates that they are above the expected level and green indicates that they are "at risk" or sufficiently below the expected level

[14] In New Zealand most students start school on their fifth birthday and have their eighth birthday while in Year 3 and eleventh birthday while in Year 6.

that their future learning in mathematics is in jeopardy. The school also used similar tables to analyze the progress and achievement of their Māori students and compare boys and girls achievement at the end of the year.

Numeracy Student Achievement Information: All Students at School X

Curruculum Levels	Numeracy Stages																		
Level 5	Stage 8																		2
Level 4	Stage 7																2		2
Level 3	Stage 6									4			6	1	4	10	4	9	19
Level 2	Stage 5		1				1	1	5	12	2	5	25	8	25	20	19	20	14
	Stage 4		1	6		4	26	9	13	20	13	31	13	18	12	13	17	13	7
	Stage 3	1	1	9	7	11	11	10	16	5	26	9	1	16	5	4	5	4	2
	Stage 2	6	14	10	17	23	7	15	8	3	4			6	3		0		
Level 1	Stage 1	5	10	1	16	7		9	3	1	0			0			1		
	Emergent	15		1	6	1	1	1											
		Year 1			Year 2			Year 3			Year 4			Year 5			Year 6		
	Time of year	Feb	Jun	Nov	Feb	Jun	Nov	Feb	Jun	Nov	Feb	Jun	Nov	Feb	Jun	Nov	Feb	Jun	Nov
	Totals	27	27	27	46	46	46	45	45	45	45	45	45	49	49	49	46	46	46
At or above expectatior		26%	63%	93%	0%	9%	59%	22%	40%	80%	4%	11%	69%	18%	59%	65%	9%	20%	50%
Cause for Concern		74%	37%	7%	52%	74%	39%	22%	36%	11%	29%	69%	29%	37%	24%	27%	41%	43%	30%
At Risk		0%	0%	0%	48%	17%	2%	56%	24%	9%	67%	20%	2%	45%	16%	8%	50%	37%	20%

Above Expectation | Expectation stage for class | Cause for Concern | At Risk

This evidence based approach has been supported by the development and publication of *Effective Pedagogy in Mathematics/Pāngarau* as part of the Ministry of Education Best Evidence Synthesis Programme[15]. The quality of this synthesis has been recognised through its publication by the International Academy of Education as part of its *Educational Practices Series*[16].

Formative Assessment in Action

The role of formative assessment or assessment for learning[17] is an integral component of the Numeracy Development Project. At the heart of the project's philosophy are teachers listening, watching, noticing and talking with students, with the information gained used to develop or modify classroom programmes based on the learning needs of the students.

For example, a teacher plans a lesson to teach a group of students how to subtract groups of ten from any three-digit number in their heads, e.g. 214 – five tens. At the start of the lesson the teacher gives the students a short activity to check their existing knowledge and notices that the students have trouble with questions like, "9 tens + 24" or "sixty three + 70". Using this

[15] www.educationcounts.govt.nz/themes/BES
[16] www.ibe.unesco.org/en/services/publications/educational-practices.html
[17] "Assessment for Learning is part of everyday practice by students, teachers and peers that seek, reflects upon and responds to information from dialogue, demonstration and observation in ways that enhance ongoing learning" Draft position paper from the Third International Conference on Assessment for Learning, Dunedin, New Zealand, March 2009.

information the teacher quickly modifies the lesson to focus on helping the students to understand and become confident with adding groups of ten to a number. The teacher first asks the students to explain how they worked out their answers, listening, watching and reflecting to work out what to do to help the students. The responses of the students guide the teacher throughout the lesson.

Formative assessment is not only used at the classroom level. The adviser who starts a "How to teach decimals" workshop by gathering feedback from the teachers about the previous workshop on teaching fractions is also gathering information that can be used in a formative way. Finding that the teachers have many questions and are confused about teaching fractions, the adviser modifies the workshop to focus on answering the teachers rather than proceeding with the predetermined plan is an example of formative assessment in action or assessment being used for learning. The adviser's decision to change the workshop is based on their knowledge that the way they were going to teach decimals is based on teachers fully understanding how to teach fractions. The adviser decided on the spot that without a full understanding there was no point in proceeding as planned as it would lead to even more confusion, best to help the teachers understand how to teach fractions before moving ahead.

Modifying the way things are done also occurs at the project level. An example of this in New Zealand was the commonly held belief that understanding place value, (hundreds, tens and ones) would not be a problem for students studying mathematics in Māori medium settings. This belief arises out of the fact that the Māori language itself assists with this understanding, for example, 27 is "rua tekau ma whitu" which literally means two tens and seven. However student achievement research findings showed that this belief was not true. This research finding resulted in a change to the Te Poutama Tau to a deliberate focus on place value. Significant improvements in student's understanding of place value were reported in the following year's research findings.[18]

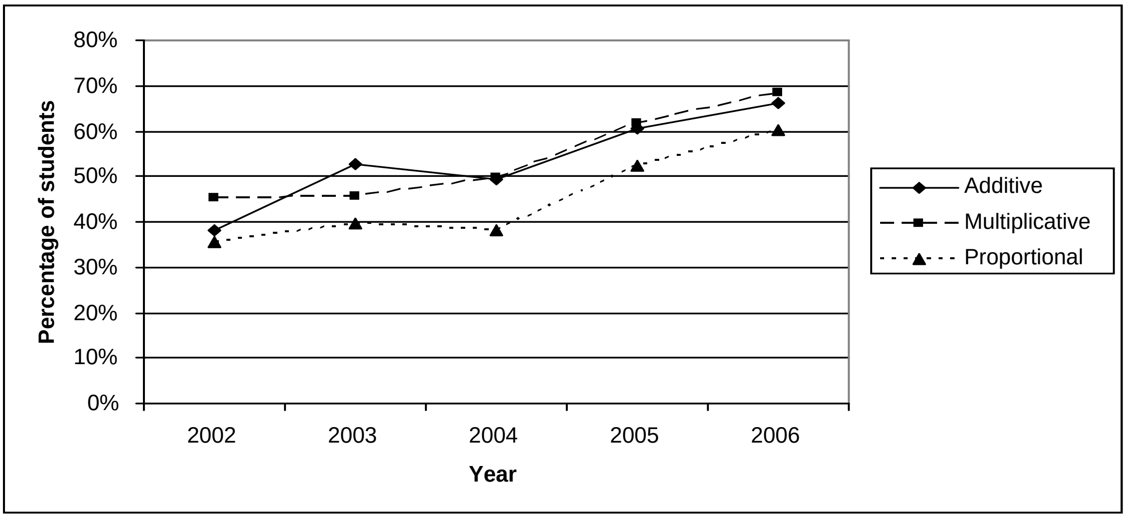
Challenges and opportunities

There is no doubt that the Numeracy Development Project and Te Poutama Tau have been effective in improving student outcomes. Overall student achievement has improved over the years of the project, and the disparities between the achievements of different ethnic groups are reducing.[19]

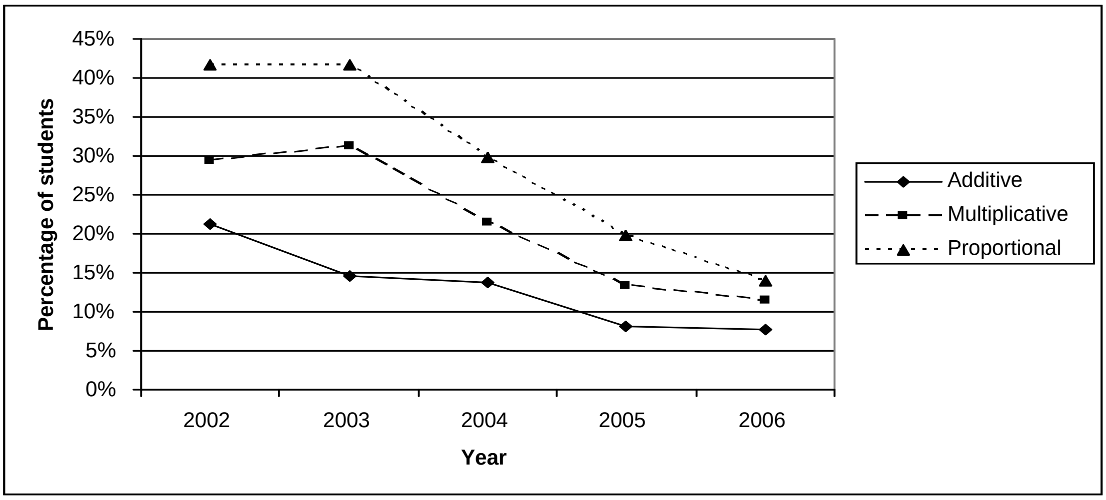
The following graphs show the improvement in the percentage of Year 6 students at or above the expected level, and the reducing percentage well below the expected level[20] at the end of the year. These are students in this study have been in classrooms with teacher who have recorded numeracy achievement information at the end of each year at school from Year 1 through to Year 6, i.e. their stage of achievement from the Number Framework. Most Year 6 students are 11 years old at the end of Year 6.

[18] Christensen (2004)
[19] Satherley, (2010)
[20] Thomas and Tagg, (2008)

Percentage of Year 6 students at or above the expected stage



Percentage of Year 6 students rated as well below the expected stage



In the ten years it has been operating, the project has provided a unique opportunity to develop an understanding about the design of powerful professional development that improves student outcomes. The pedagogical tools of the number framework, the diagnostic interview, and the strategy teaching model are critical elements of the professional development design. The integration of these elements ensures a focus on the core ideas of improving teacher knowledge of mathematics, enhancing understanding of how students learn mathematics, assessment capability and enhancing understanding of how to represent mathematical concepts.

From a system perspective, the outcomes of improving the quality of teaching and student achievement through an unrelenting focus on the core of teaching practice – curriculum, assessment, and pedagogy – in the context of the teacher's own classroom has provided the opportunity to learn about scaling up professional development provision, while maintaining the capacity to effect deep and consequential change. This large-scale case study in mathematics education is evidence that all students benefit when pre-service teacher educators, researchers, school advisers, teachers, and policy makers work together for educational reform, using what is known from research to design and deliver powerful professional development, (Higgins & Parsons, 2009).

Currently New Zealand is introducing National Standards in mathematics. The National Standards in mathematics themselves and the implementation plan have been designed to build on the Numeracy Development Project. A solid platform has been laid over the ten years of the project so that National Standards can be integrated and used to further develop and improve mathematics education in New Zealand. By continuing this approach of aligning and interconnecting all parts New Zealand is moving closer to having all students leave school with a positive attitude towards mathematics, coupled with an understanding and ability to use mathematics effectively whenever needed.

References

Alton-Lee, A. (2003). Quality teaching for diverse students in schooling: Best evidence synthesis iteration. Wellington: Ministry of Education.

Anghileri, Julia. (2001). Contrasting approaches that challenge tradition. In J. Anghileri (Ed.)(2001) Principles and practices in arithmetic teaching (pp.4-14). Buckingham: Open University Press.

Anthony, Glenda & Walshaw, Margaret. (2002). Swaps and Switches: Students' understandings of Commutativity. In B. Barton, K. Irwin, M. Pfannkuch, & M. Thomas (Eds.)(2002). Mathematics Education in the South Pacific: Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia,(pp.91-99). Sydney: MERGA

Anthony, G., & Walshaw, M. (2007). Effective pedagogy in mathematics/pāngarau: Best evidence synthesis iteration. Wellington: Ministry of Education.

Anthony, G., & Walshaw, M. (2009). *Effective pedagogy in mathematics*. Educational Practice Series – 19. International Academy of Education & International Bureau of Education Paris. UNESCO.

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. The Elementary School Journal, 93, (4), 373-97.

Black, P. J., & Wiliam, D. (1998). Assessment and classroom learning. Assessment in Education: Principles, Policy and Practice, 5 (1), 7–74.

Bobis, J. (1996). Visualisation and the development of number sense with kindergarten children. In J. Mulligan& M. Mitchelmore (Eds). Children's number learning. Adelaide: The Australian Association of Mathematics Teachers, 1996, pp. 17-33.

Boulton-Lewis, G., Wilss, L. & Mutch, S. (1996). Representations and Strategies for subtraction used by primary school children. Mathematics Education Research Journal, 8 (2), 137-152.

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction, The Elementary School Journal, 97 (1), 3-20.

Christensen, I. (2004). Exploring Issues in Mathematics Education, An Evaluation of Te Poutama Tau 2003. Wellington: Ministry of Education.

Clark, F. B., & Kamii, C. (1996). Identification of multiplicative thinking in children in grades 1-5. Journal for Research in Mathematics Education, 27 (1), 41-51.

Clements, D. H. (1999). Subitizing: What is it? Why teach it? Teaching Children Mathematics, March, 400-405.

Cobb, P., & Wheatley, G. (1988). Children's initial understandings of ten. Focus on Learning Problems in Mathematics, 10(3), 1-26.

Fuson, K. C. & Briars, D. J. (1990). Using a base-ten blocks learning/teaching approach for first- and second-grade place-value and multidigit addition and subtraction. Journal for Research in Mathematics Education, 21 (3), 180-206.

Higgins, J., & Parsons, R. (2009). A Successful Professional Development Model in Mathematics: A System-Wide New Zealand Case. Journal of Teacher Education, 60 (3), 231-242.

Hughes, P. (2002). A model for teaching numeracy strategies. In B. Barton, K. Irwin, M. Pfannkuch, & M. Thomas (Eds.)(2002). Mathematics Education in the South Pacific: Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, (pp.350-357). Sydney: MERGA

Irwin, K., & Ell, F. (2002). Visualising and the move from informal to formal linear measurement. In B. Barton, K. Irwin, M. Pfannkuch, & M. Thomas (Eds.)(2002). Mathematics Education in the South Pacific: Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, (pp.358-365). Sydney: MERGA

Irwin, K., (2003). Multiplicative Strategies of New Zealand Secondary School Students. In Pateman, N. A., Dougherty, B. J., & Zilliox, J. T. (Eds.)(2003). Proceedings of the 2003 Joint Meeting of PME and PMENA, CRDG, College of Education, University of Hawai'i

Lamon, S. J. (1996). The Development of unitizing: Its role in children's partitioning strategies. Journal for Research in Mathematics Education, 27 (2), 170-93.

Lampert, M. (1989). Choosing and using mathematical tools in classroom discourse. Advances in Research on Teaching, 1, 223-64.

Maclellan, E. (1997). The importance of counting. In I. Thompson (Ed.). Teaching and learning early number (pp.33-40). Buckingham, UK: Open University Press.

McIntosh, A. (1996). Mental computation and number sense of Western Australian students. In J. Mulligan & M. Mitchelmore, (Eds). Children's Number Learning (pp.259-276). Adelaide: Australian Association of Mathematics Teachers.

Mulligan, J. T. and Mitchelmore, M. C. (1997). Young children's intuitive models of multiplication and division. Journal for Research in Mathematical Education, 28 (3), 309-330.

Pirie, S., & Kieren, T. (1989). A recursive theory of mathematical understanding. For the Learning of Mtahematics, 9(3), 7-11.

Pitkethly, A. & Hunting, R. (1996). A review of recent research in the area of initial fraction concepts. Educational Studies in Mathematics, 30, 5-38, 1996.

Ritchie, G. (1991). Does mathematics equipment help? Interviews with children using equipment. SAME Papers, 79-101

Satherley, P. (2010). Performance of the New Zealand education system: What we know from research, statistics and international studies. Working Paper 11, Research Division, Wellington: Ministry of Education

Sowder, J. (1997). Place value as the key to teaching decimal operations. Teaching Children Mathematics, 448-53, April.

Steffe, Leslie. (1994). Children's multiplying schemes. In G. Harel & J. Confrey (Eds.) The development of multiplicative reasoning in the learning of mathematics (pp. 3-39). Albany,NY: SUNY.

Thomas, N. (1996). Understanding the number system. In J. Mulligan & M. Mitchelmore (Eds.). Children's Number Learning, (pp. 89-106). Adelaide, South Australia: MERGA/AAMT.

Thomas, G., and Tagg, A. (2008). What do the 2002 school entrants know now? Findings from the New Zealand Numeracy Development Projects 2007 (pp. 5–15). Wellington: Learning Media.

Thomas, Gill, Tagg, Andrew, & Ward, Jenny. (2002). Making a difference: The Early Numeracy Project. In B. Barton, K. Irwin, M. Pfannkuch, & M. Thomas (Eds.)(2002). Mathematics Education in the South Pacific: Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, (pp.49-57). Sydney: MERGA.

Thompson, I. (1997). Developing young children's counting skills. In I. Thompson (Ed.) Teaching and learning early number (pp.123-132). Buckingham, UK: Open University Press.

Timperley, H., Wilson, A., Barrar, H., & Fung, I. (2007). Teacher professional learning and development: Best evidence synthesis iteration. Wellington: Ministry of Education.

Timperley, H. (2008). *Teacher professional learning and development*. Educational Practice Series – 18. International Academy of Education & International Bureau of Education Paris. UNESCO.

Wright, R. (1998). An overview of a research-based framework for assessing and teaching early number. In C. Kanes, M Goos, & E. Warrens (Eds.), Proceedings of the 21st Annual Conference of the Mathematics Education Group of Australasia (Vol. 2, pp. 701-708). Brisbane, Australia: Griffith University.

Yackel, Erna (2001). Perspectives on arithmetic from classroom-based research in the United States of America. In J. Anghileri (Ed.)(2001) Principles and practices in arithmetic teaching (pp. 15-31). Buckingham: Open University Press.

Young-Loveridge, J. (1999). The acquisition of numeracy. Research Information for Teachers, 1.

Young-Loveridge, Jenny & Wright, Vince. (2002).Validation of the New Zealand Number Framework. In B. Barton, K. Irwin, M. Pfannkuch, & M. Thomas (Eds.)(2002). Mathematics Education in the South Pacific: Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, (pp.722-729). Sydney: MERGA

Learning Progressions: Informing and Supporting Instruction and Formative Assessment

Erma Anderson, US Office of Overseas Schools

March 2010

Abstract

"Formative assessment is a process used by teachers and students during instruction that provides explicit feedback to adjust ongoing teaching and learning to improve students' achievement of intended instructional outcomes."(McManus, S. (2006). Formative assessment is a critical component of a balanced assessment system. Classroom-based formative assessments provide evidence of student thinking. The evidence collected enables teachers to differentiate instruction based on students' cognitive strategies rather than on incorrect answers. Teachers practicing formative assessment ask students to perform tasks, explain their reasoning, and prove their solutions. Teachers who engage in formative assessments give continual, explicit feedback to students and assist them in answering the questions: Where am I going? Where am I now? How can I close the gap between the two?

This monograph will share the program AERO SAW that uses teacher and student artifacts to reflect on assessment practices. Research indicates that formative assessment, if well-designed and implemented correctly, is an effective strategy for enhancing student learning. Research concludes that compared to other interventions, formative assessment has the greatest impact on learning gains and is more cost-effective. We will share the journey, the process, the challenges and how they were addressed of a group of schools implementing the AERO SAW model for examining assessment practices.

Introduction

Current discussions about teacher learning stress the potential advantage of embedding that learning in aspects of teachers' practice. (Ball & Cohen, 1999; Cochran-Smith & Lytle, 1999; Lampert & Ball, 1999) Organizing teacher learning around the study of artifacts of practice, student tasks, student work, and student feedback, is one way to embed the learning into practice. Little (1999) contends that " one of the most powerful and least costly occasions of teacher learning is the systematic, sustained study of student work, coupled with individual and collective efforts to figure out how that work results from the practices and choices of teaching" (p.235)

Black and Wiliam (1998b) define assessment broadly to include all activities that teachers and students undertake to get information that can be used diagnostically to alter teaching and learning. Assessments become formative when the information is used to adapt teaching and learning to meet student needs. Formative assessment is tightly linked with instructional practices.

There is considerable evidence that assessment, when practiced effectively, can improve student learning (Black & Wiliam, 1998). One of the most powerful research- based strategies for linking assessment to improved instructional practice is teacher collaboration on analyzing student tasks, student responses, and teacher feedback to student responses. Little et al. (2003) found that teachers who engaged consistently in such discussions were able to:

- assess student performance more consistently, effectively confidently, and fairly;
- build common knowledge about curriculum expectations and levels of achievement;
- identify strengths and areas for growth based on evidence of student learning;
- adjust and acquire new learning by comparing one's thinking to that of another student or teacher;
- share effective practices to meet the needs of all students, monitor progress, and celebrate growth.

Teachers need to consider how their assessment practices (classroom activities and assignments) support learning goals, provide students opportunities to communicate what they know, and how they use this information to improve teaching and learning. Opportunities for professional dialogue about assessment practices bring coherence to those practices, while promoting a climate of inquiry that supports student learning, and challenges teachers to focus future instruction on specific learning outcomes.

The United States State Department Office of Overseas Schools (A/OS) assists 196 schools with an enrollment of about 124,000 students in 136 countries. The purposes of the assistance is to help these independent schools fulfill their mission of providing quality education for children of dependents of American citizens carrying out the programs and interests of the U.S. Government abroad and to demonstrate to foreign nationals the philosophy and methods of American Education.

Background

In the 1990's, the Near East South Asia Council of Overseas Schools (NESA), in conjunction with the US Department of State **Office of Overseas Schools** Project **AERO**, used the NCTM Standards to develop a set of mathematics standards. The project was funded by the Overseas Schools Advisory Council (OSAC). The AERO Mathematics Standards followed the model of standards that was then being developed typically organized into grade spans. The goal of this model was to allow curriculum flexibility with the idea that a student would understand the concept, "By the End of Grade…". The general guide for placement of standards at each grade was the perceived beliefs within the content community of each school about when and how the big ideas of mathematics unfolded.

AERO (American Education Reaches Out), a project supported by the U.S. State Department's Office of Overseas Schools (A/OPR/OS) and the Overseas Schools Advisory Council, provides overseas schools with standards for curriculum consistency and for stability of curriculum across grades K-12. AERO helps overseas schools implement and sustain standards-based curricula that is in alignment with research-based trends in the development of curriculum worldwide, and particularly with standards-based efforts in the U.S. www.projectaero.org

Even though meeting standards was the ultimate goal of instruction in our schools, the AERO standards, like most state standards, did not provide a clear picture of what learning was expected. Since the expected learning was not clear, teachers needed elaboration before they could use the Standards documents as a basis for instruction and assessment. This limited use for planning instruction and assessing learning also hindered schools in developing a comprehensive coherent curriculum, essential in schools where the mobility of students and teachers is very high.

In 2003, Project AERO began a second effort. AREO SAW, with an OSAC grant to NESA and A/OS support. AERO SAW, in collaboration with CBE, was a two-year project, which examined student work. A select group of schools in the region piloted the two-year project, AERO SAW, to examine student work for concrete evidence of what the teacher intended and what the student learned. The process provided a structured format for teachers to examine the artifacts of teacher practice and student learning. Central to the project is the reflection and questioning that guides small critical friend groups of teachers in an analysis of their assignments and student work resulting from those assignments.

AERO SAW: The Process

Schools Around the World (SAW) was a program of the Council for Basic Education. It was multinational professional development model that used world-class standards as the basis for improving student achievement. It gives science and math teachers from around the world the opportunity to use student work to improve achievement by examining and reflecting upon their own teaching practice. In the United States, Schools Around the World worked through a combination of both in school workshops and online seminars.
http://cct2.edc.org/saw2000/frontfrm.htm?saw_ov.htm

AERO **SAW** uses a structured format (See Appendix A) to help teachers examine the artifacts of teacher practice and student learning. It is a process for linking instruction and formative assessment to improve student learning. Through **critical friend** discussions using the structured protocol, Evidence to Excellence (E2E) Placemat (See Appendix B) small groups of teachers discuss and reflect on student tasks, student work, teacher assessment and feedback of the work.

A teacher shares a task he/she has given to students. Using the protocol, the critical friends reflect on their observations. Discussions center on how the task related to the intent of the standard(s)/benchmark(s), the prior knowledge required to be successful, the clarity of the language, the rigor of the task, and what would be sufficient evidence of student learning.

The sharing teacher then presents three samples of student work. Critical friends discussions center on the students reasoning skills and evidence of students learning. Not knowing how the sharing teacher has assessed the student work, critical friends share their assessment.

The sharing teacher then shares how he/she had assessed the work and the feedback he/she provided each student. Critical friends then reflect on the teacher assessment and feedback. (See Appendix C)

AERO SAW: The Model

The quality of work in professional learning communities depends on the quality of collaboration that is embedded into a school's culture. Michael Fullan states that "collaborative cultures, which by definition have close relationships, are indeed powerful, but unless they are focusing on the right things they may end up being powerfully wrong."

So what are the "right things"? DuFours, Barth, Schlechty, etc. have determined that schools which truly embrace a mission of learning for all, there is a focus on four critical questions:

- Ø What is it we want all students to learn?
- Ø How will we know when each student has mastered the intended learning?
- Ø How will we respond when a student does not master the intended learning?
- Ø How will we respond when a student has already learned it? (DuFours. Professional Learning Communities At Work)

E2E conversations provide schools an opportunity to work in professional learning communities where educators can engage in conversations about the many factors that affect student achievement.

Near East South Asia Council of Overseas Schools (NESA) is a non-profit voluntary association of more than 90 private independent international schools in the Near East and South Asia. Regular member schools follow an American/ International college preparatory curriculum and typically serve students of more than four dozen nationalities. http://www,nesacenter.org

The AERO SAW model is a combination of face-to-face and online collaborations. Research studies have demonstrated that the best professional development is not face-to-face only or online only, it's both. "It's a widely held misconception that any form of online learning is second best to any form of face-to-face learning. What research shows us is that online learning and face-to-face learning complement each other in interesting ways. Some people who are silent in face-to-face professional development sessions find their voice in online interactions, for a variety of reasons. Online learning can also extend time, which is perhaps the most precious resource that teachers have, because it allows them to do professional development when they want, where they want. So it has some strengths that are a really good complement to face-to-face professional development." (Dede, C. 2006)
AERO SAW, a NESA project, began with teams of teachers from overseas schools that included all regions of the world. The preconference session provided opportunities for school teams to become familiar with the E2E process and to provide the tools needed to return to their schools to facilitate small group discussions. During the school year, online support was provided and teams were reconvened to discuss successes and challenges at the NESA

Spring Conference. Discussions of teacher and student artifacts combined the discussions of the subject matter, student thinking, and instruction seamlessly. The discussions initiated questions about the context of the lesson, prior knowledge, evidence of student learning, and feedback to

move students forward. A follow-up program involved fewer schools and included face-to-face meetings at several NESA conferences. School-based training was also provided as well as three-week online seminars. Project AERO provided summer training in AERO SAW (E2E) over a period of three years.

Rethinking Mathematics Standards

Formative assessment is an ongoing process of collecting evidence of student learning and using that evidence to identify next steps for learning. The elements of formative assessment include:

- Being specific about what we want students to learn
- Eliciting the evidence of student learning to identify gaps between current and desired performance
- Interpreting the evidence to identify next steps for learning.
- Providing feedback to students for reflection on their learning and generating next steps

E2E focused on the goals of student tasks, the evidence of student learning, and interpreting that evidence to provide next steps for students and teachers. However, the AERO Mathematics Standards had followed the model of standards development at that time and standards were organized by grade spans. These did not provide a clear picture of what learning was expected. Since the expected learning was not clear, it was difficult for teachers to determine where students were relative to the standards. By its very nature, learning involves progression and it was imperative teachers understand the pathways along which students were expected to progress before they could make decisions about what the next steps in learning should be. Without an understanding of the continuum of learning for the domain, conversations were restricted to the task given to students to meet the goal of the lesson and evidence of success.

> “Placement of the standards should reflect the grade level at which mastery is expected, and standards should not be repeated from year to year.” National Mathematics Advisory Panel

To address this challenge and to provide greater consistency to the mathematics curriculum in A/OS schools, AERO developed a Framework for Mathematics (www.projectaero.org). The National Council of Teachers of Mathematics Curriculum Focal Points (NCTM, 2006) and the international benchmarking guided the development of the document. The Focal Points specify for each grade level the most important mathematical ideas that a student needs to understand in-depth for future mathematics learning. The K-8 document was designed so that teachers could view progression points of the learning continuum across grade levels. This articulation of learning progressions described a pathway of learning that would assist teachers in planning instruction, tying formative assessment to the expected learning, and pinpointing where students’ learning was on the continuum. Identifying where each student is on the continuum of learning in the various domains of mathematics has been facilitated with the use of Adaptive assessments. The results provide many practical applications for teaching and learning.

Rethinking Professional Development for Teachers

Seventy-five A/OS schools use the North West Evaluation Association (NWEA) Measure of Academic Progress (MAP) assessment to gather this information. MAP is a computerized adaptive math test that reflects the instructional level of each student and measures academic growth over time, independent of grade level or age. MAP is aligned to the AERO Framework

The purpose of formative assessment is to adjust teaching based on evidence about learning so that students can close the gap between where they are now and the desired learning goal. If teachers are not clear about next steps for moving student learning forward, then the promise of formative assessment to improve student learning is greatly diminished.

To know what to do next in response to formative assessment evidence, teachers need a clear understanding of how learning progresses. However, learning progressions, by themselves, are not sufficient. A deep knowledge of the content represented in the learning progression is also needed. Effective formative assessment, requires optimization of mathematics knowledge, pedagogical content, assessment knowledge, and knowledge of students' previous learning (Heritage, 2007). If teachers are clear about these aspects, they will be better prepared to respond to them when they show up in formative assessment. A recent study, Heritage et al. (2009) found that teachers had the skills to use data and draw inferences but fell short with respect to planning "the next instructional steps" (Heritage, 2009,p. 31).

To help all students learn mathematics, teachers need to understand the mathematics they teach and, when possible, to understand it in several ways as well as several kinds of knowledge about learning. Teachers need to see how ideas connect across fields and to everyday life. This kind of understanding provides a foundation for pedagogical content knowledge that enables teachers to make ideas accessible to others (Shulman, 1987). Acquiring this sophisticated knowledge and developing this practice is different from what most teachers have experienced as students and it requires providing learning opportunities that are more powerful than simply reading and talking about mathematics (Ball & Cohen, 1996). Teachers learn best by studying, by doing and reflecting, by collaborating with other teachers, by looking closely at students and their work, and by sharing what they observe. This kind of learning cannot occur in environments divorced from practice or in school classrooms divorced from knowledge about how to interpret practice. Good settings for teacher learning must provide lots of opportunities for research and inquiry, for trying and testing, for talking about and evaluating the results of learning and teaching. The combination of theory and practice (Miller & Silvernail, 1994) occurs most productively when questions arise in the context of real students and work in progress and where research and disciplined inquiry are also at hand.

The depth of teacher knowledge of K-6 teachers, particularly as it relates to teaching mathematics, is an issue and "too many professional development programs fall into the category of 'tips for teachers' rather than extending knowledge about how learning develops in a domain that can be applied and enriched as teachers acquire experience teaching" (Heritage, 20).

To support teachers as they develop their understanding of learning progressions and the mathematics content, AERO is piloting a two-year, content-oriented professional development experience for K-8 teachers in four NESA schools. The project: *Meeting The Challenges of the 21st Century (MCI[2]): Transforming Teacher Learning to Student Learning* will engage teachers in experiential activities designed around the AERO Mathematics Curriculum Framework.

The Project has three foci:

1. Building mathematical content knowledge and pedagogical expertise for the teachers of mathematics and subsequently improving student understanding of mathematics.
2. Creating and building a network of teachers who will model effective teaching, sharing mathematical content knowledge in their schools, guiding and contributing to decisions about district (school) curriculum and professional development.
3. Creating a supportive family of international educators drawn together by the common experience. The ultimate goal is for this network to build its own capacity to facilitate similar conversations in their own schools.

"A focused, coherent progression of mathematics learning, with an emphasis on proficiency with key topics, should become the norm in elementary and middle school mathematics curricula. Any approach that continually revisits topics year after year without closure is to be avoided." National Mathematics Advisory Panel

Discussions on specific learning progressions will occur in each school, a week in the fall and a week in the spring. The conversations will focus on Making Sense of Number Sense, Algebraic Thinking in the K-5 Curriculum, Data Analysis in the K-5 Curriculum, and Problem Solving. MCI[2] is designed to help K-5 teachers revisit and extend their mathematical knowledge and build it into this specialized kind of knowledge needed for effective mathematics teaching and learning.

Rethinking: The Mathematics Curriculum

Another challenge facing teachers was the curriculum. The curriculum of most schools is a textbook or it is organized around scope and sequence charts that specify procedural objectives to be mastered at each grade. Usually, these are discrete objectives and not connected to each other in a larger network of organizing concepts. Most textbooks cover a wide array of topics, not always organized in a logically connected way leading to a "mile wide and an inch deep" curriculum (Schmidt, McKnight & Raizen, 1997:1)

Curricula organized into "units" of instruction around particular topics present better opportunities for instructional planning and formative assessment. When 'units' are described in terms of a core concept or "big idea" and supporting sub-concepts teachers are more easily able to map formative assessment onto these learning goals. However, this approach to organizing content has its own set of drawbacks. Units are often not connected to each other in a coherent vision for the progressive acquisition of concepts and skills, and therefore limit teachers' ability to see how learning develops in a specific domain (Heritage, 2009). Schools participating in MCI[2] are building their units around the big ideas in mathematics.

Conclusion

When AERO SAW was first introduced to A/OS Schools, the process gave schools an opportunity to engage in conversations about formative assessment and teacher and student learning. However, as teachers engaged in the process, it became clear that if formative assessment was to be an integral part of the professional practice in schools and if there was to be rich conversations about student learning, our standards must be more clearly defined.

Learning progressions have been a powerful model for re-envisioning our standards, assessments, instruction, curricula, instruction, and professional development in mathematics in a way that is grounded in current research on mathematics learning.

Learning Progressions offer a clear picture of where the students have been and where they are headed. They can be used to map and align K-12 curriculum, guide resource selection, and as jumping off points for professional conversations about methods and approaches to improve mathematics teaching and learning. Learning progressions have the potential to expand and enhance the AERO SAW E2E conversations and to provide teachers greater opportunity to make instructional decisions grounded on the learning research.

References

Ball, D.L. & Cohen, D. K. (1999). Developing practice, developing prctitioners: toward a practice-based theory of professional development. In L. Darling-Hammond & G. Skyes (Eds.), *Teaching as the learning professional: Handbook of policy and practice*. (pp. 3-32). San Francisco: Jossey-Bass.

Barth, R. (1991). Restructuring schools: Some questions for teachers and principals. *Phi Delta Kappan, 73*(2), 123–128.

Black, P. and Wiliam, D. (1998b). Inside the black box: Raising standards through classroom assessment. Phi Delta Kappan, 80 (2): 139-148. (Available online: http://www.pdkintl.org/kappan/kbla9810.htm.)Brophy,J.,&Good,T. (1986). Teacher behavior and student achievement. In M. Wittock (Ed.),Handbook of research on teaching (pp.328-375). New York:Macmillan

Cochran-Smith, M., & Lytle, S. (1999). Relationships of knowledge and practice: Teacher learning in community. In the series, Review of Research in Education, 24, 249-305. Washington, DC: American Educational Research Association.

Davydov, V.V. (Ed.). (2008). Problems of developmental instruction: A theoretical and experimental psychological study (P. Moxhay, Trans.). New York: Nova Science Publishers.

Dede, Chris, **Online Professional Development for Teachers: Emerging Models and Methods** *(Harvard Education Press, 2006),*

DuFour, R., DuFour, R., Eaker, R., & Many, T. (2006). *Learning by doing: A handbook for professional learning communities at work.* Bloomington, IN: Solution Tree.

Fullan, M. (2006). Leading professional learning. *School Administrator*, *63*(10), 10-14.

Gong, B. (2007, June). *Learning progressions: Sources and implications for assessment*. Presentation at the CCSSO, Large-Scale Assessment Conference, Nashville, TN.

Heritage, M. (2008). Learning progressions: Supporting instruction and formative assessment. Washington, DC: Council of Chief State School Officers. Retrieved December 2, 2009, fromhttp://www.ccsso.org/publications/details.cfm?PublicationID=366

Heritage, M., Kim, J., Vendlinski, T., & Herman, J. (2009). From evidence to action: A seamless process in formative assessment? Educational Measurement: Issues and Practice, 28(3), 24–31

Lampert, M. & Ball, D. (1999). Aligning teacher education with contemporary K-12 reform visions. In L. Darling-Hammond and Gary Sykes, eds., Teaching as the Learning Profession: Handbook of Teaching and Policy . NY: Jossey-Bass.

Little, J.W. (1993). Teachers' professional development in a climate of education reform. Education Evaluation and Policy Analysis, 15(2), 129-151.

Little, J. W. (2003). Inside teacher community: Representations of classroom practice. Teachers College Record 105(6): 913-945.

Marzano, R. (2003). *What works in schools: Translating research into action*. Alexandria, VA: ASCD

Marzano, R.J., Pickering, D.L., Pollack, J.E., (2001). Classroom instruction that works, Alexandria, Va: ASCD.

McManus, S. (2006). "Attributes of Effective Formative Assessment." Accessed March 20, 2009 from The Council of Chief State School Officers website.

National Council of Teachers of Mathematics. (2006). *Curriculum Focal Points to Focus Math Curricula.* Retrieved from: http://www.nctm.org/news/releases/2006_09ctp.htm.

Nation Research Council (2003). *Adding It Up: How Children Learn Mathematics*. Mathematics learning Study Committee, Kilpatrick, Swan, & Findell (Eds.) Washington, D.C.: National Academy Press.

National Research Council. (2001). *Knowing What Students Know: The science and design of educational assessment*. Committee on the Foundations of Assessment. J. Pelligrino, N. Chudowsky, & R. Glaser (Eds.), Board on Testing and Assessment, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.

Shulman, L.S. (1986). Those who understand: Knowledge growth in teaching. Educational Researcher, 15(2), 4-14

Section II

Replicating AERO SAW

It's very difficult for teachers to critically examine the quality of their assignments, assessments, and feedback in isolation. At the same time, critical, constructive feedback from peers is not often welcomed. But without critical feedback, it is difficult to improve teaching effectiveness. The E2E protocol is a set of guidelines to promote these meaningful conversation about teaching practice. Having a structured agenda provides a safe environment for a teacher to share their students' work and reflect on the probing questions of critical friends about the quality of an assignment, the resulting student work, and teacher feedback. The protocol structure helps the group focus for a set period of time and gives them permission to ask challenging questions of each other and builds in time for the presenting teacher to listen and reflect back. The use of a protocol is one way to make the most of the small bits of time that teachers have to engage with their peers. The point is not following the protocol exactly but to have in-depth, meaningful conversations about teaching that lead to improvements in practice.

How can a collaborative reflective process be implemented by others? Evidence to Excellence can flourish in small learning communities within schools if it is approached in an organized manner. The protocol, its implementation, and application along with a particular focus for the work (e.g., reflection on intellectual quality of the tasks, assessments, student work, units) must be approached carefully and purposefully if teacher learning is to occur,

Form teacher groups of 6-10 people, add a well trained facilitator armed with a good protocol and you are on your way to a successful professional experience. However, creating an effective and sustained program is more complicated than described. If the process is to be implemented, trained facilitators are needed to lead such groups.

Trainings:

The first step in a successful E2E experience is facilitator training. Training facilitators is critical to the success of the program. Teachers usually work in isolation, rarely discussing with other teachers what works or doesn't work in their classrooms. Collaborative groups provide a safe, non-judgmental place where teachers can have their work sympathetically critiqued. A critical friend partners you on the journey of reflection and learning. Yet, while their main purpose is to provide support, they are not afraid to confront, with questions, in order to stretch thinking and help one become more reflective on their practice. Good facilitation of the process is key to the success.

The AERO SAW materials can easily be adopted by any entity interested in examining teacher assignments, student responses to those assignments, and the feedback given students on those assignments. All of the materials and protocol are available at no cost. To implement the program would require training facilitators in the process. Face to face and online seminars can also be made available.

Seminars

AERO SAW was a two year plan with Year 1 workshops 1 and 2 completed in the first face to face and workshops 3 and 4 completed in the second face to face workshop. The same process was used in Year 2. In each seminar, discussions used student and teacher work to ground the conversations.

Seminar 1: Introduction to looking at student work, focusing on standards and inquiry.
Online Seminar 1: Interactive collaboration to reflect on the process of learning through inquiry.

Seminar 2: "Meatiness" of the assignment
Online Seminar 2: Interactive collaboration to analyze and reflect on the rigor of assignments.

Seminar 3: Formative Assessment
Online Seminar 3: Interactive collaboration to reflect on the assessment of student work and feedback given students.

Seminar 4: Knowledge vs. understanding
Online Seminar 4: Interactive collaboration to analyze and reflect on assessing for understanding.

Seminar 5: Intellectual quality of student work
Online Seminar 5: Interactive collaboration to analyze and reflect on the intellectual quality of student work.

Section III

Materials

Materials needed for successful implementation are the E2E protocol, Facilitator training materials, workshop and online seminar materials. All materials are in English.

AERO SAW is focused on improving student learning. An important part of this work is an improvement in instruction. Meaningful professional development can not take place without rigorous standards and a means to assess student learning. Examining teacher and student artifacts is useful in making educational decisions regarding student achievement. To do this effectively, teachers need to have in mind **a continuum of how learning develops in mathematics** so that they are able to locate students' current learning status and provide feedback and decide on pedagogical action to move students' learning forward. Without this understanding of the learning continuum, the conversations focus only on the task and student product.

Effective implementation of E2E requires the following elements
(1) establishing critical friends groups with exemplary facilitators,
(2) providing administrative support, and

(3) creating relevance to context and curriculum.

Establishing critical friends groups with exemplary facilitation, a true learning community of practice is key to engaging teachers in the discussions of tasks, assessments, and feedback. The importance of active administrative support for teachers' planning and collaboration is critical. Without dedicated time, critical friends groups cannot be sustained. Administrators' support and explicit expectations are a key element to negotiating the logistics of school schedules and teachers' competing time commitments and priorities. Examining teacher and student artifacts must focus on the tasks from the teachers' classroom. Research confirms that student work from a teacher's own classroom is a critical source of evidence for learning how well a lesson was taught, what improvements are needed, and how to improve student learning.

An analysis of student work on a particular topic helps teachers to differentiate instruction so that all students in the classroom can master the concepts being taught. It provides the tangible bridge between students and teachers and provides concrete, direct evidence of what the teacher intended and what the student learned from assignments. Student work is the data that provides crucial and telling information about a classroom, and it is the focus of AERO SAW's Evidence to Excellence (E2E) process.

1
Monograph Learning Progressions: Supporting Instruction and Formative Assessment

Appendix A: E2E Process

Feedback

Reflection

1. How **clear** was the language?
2. Does the assignment provide students an opportunity to work with significant ideas and relationships that are in the **standards**?
3. How does the assignment stimulate **higher order** thinking and discussion?
4. What **evidence** would you use to determine if the student understood the content of the lesson?

What adjustments will you make to the lesson and assignment ?

Assignment

1. What is the evidence that the student used good thinking and reasoning skills in completing the assignment?
2. How does the student connect the mathematics/science they were learning to the real world?
3. What is the evidence that the student achieved the goal of the lesson?
4. How would you assess for evidence of student learning?

What adjustments will you make to instruction ?

Student Work

- Does the assessment of the student work fairly reflect the objectives of the assignment?
- How does your assessment of the work compare with the teacher's assessments?
- Does the assignment provide an opportunity to pinpoint areas of student weakness in content and thinking ability that need more intensive practice?
- What kind of feedback can be given to students?

What **interventions** should be considered to help students who do not yet meet expectations? Who meet expectations? Who exceed expectations?

Assessment

Appendix B: Evidence To Excellence; Looking At Student Work Process Placemat

Step # 1

The sharing teacher shares **a copy of the question or task** and explains the context of the lesson.

Facilitator Questions:

1. How **clear** is the language?
2. Does the assignment provide students an opportunity to work with significant ideas and relationships that are included in the district's **standards**?
3. How does the assignment **actively engage** students in constructing their own knowledge?
4. How does the assignment stimulate **higher order** thinking and discussion?
5. What **evidence** would you use to determine if the student understood the goal of the task?

Step # 2

The sharing teacher shares **three samples of student response to the question or task**, WITHOUT identifying teacher evaluations of the work. The other members of the group analyze the student work samples and provide their assessment of each sample of work.

Facilitator Questions:

1. What is the evidence that the student used **good thinking and reasoning skills** in completing the assignment?
2. What is the evidence that the student **achieved the goal** of the task?
3. How would you **assess** the work? Which piece of student work would exceed your expectations? Meet your expectations? Not yet meet your expectations?

Step # 3

The sharing teacher presents their **assessment** of the student work.

Facilitator Questions:

1. How did the teacher's assessment **reflect the objectives** of the assignment?
2. How does the teacher's feedback provide students with an opportunity to grow?
3. Why do you **agree or disagree** with the teacher's assessment?

Step # 4

Both sharing teacher and colleagues **reflect** on the question (task) and student responses.

Facilitator Questions:

1. What **interventions** should be considered to help students who do not yet meet expectations?
2. What **interventions** should be considered to help students who do not yet meet expectations?
3. What additional evidence is needed to make this collection of student work a more complete picture of student understanding of the concepts addressed?

Appendix C: Evidence to Excellence Process

Looking collaboratively at student and teacher work is a key focus of Evidence to Excellence as we work to improve student learning. Looking at student work enables participants to understand what students know and are able to do; align curriculum with district/state standards, assess academic growth over time; and design instructional practices to reach all students.

Evidence to Excellence provides a protocol, a structured format that helps participants engage in the process of collaboratively analyzing and discussing teacher and student work. The protocol helps to create a safe climate for sharing work and looking at it from multiple perspectives.

Teams: Each study group **consists of 4 to 6 people**

Grade level teams
Discipline based teams
Vertical teams

Teams need to work together over time in order to build trust that sustains open and critical conversation. It may take some teachers several months before they begin to feel truly comfortable with either showing their own work or providing critical feedback to colleagues. Teams should establish norms for the group.

Time: To foster a thoughtful and reflective discussion, the protocol requires between 75 to 90 minutes. In many cases, the time frame can be altered to accommodate the time limits of the school day.

Facilitator: Team members take turns facilitating. The role of the facilitator is to support the group's thinking and learning. Although they may participate in the discussion, they often serves best by listening and using their questions and comments to refocus the group, broaden the discussion, or summarize several points. They are responsible for creating a sense of community that values all ideas and comments and gives all individuals an opportunity to speak, A facilitator keeps the group focused, keeps the process moving along. It is critical that the concept of critical friends is kept

Presenting Teacher: At each session, one teacher agrees to bring a case of student work (described below) to share with the group. It is important teachers take turns bringing student and teacher work to share. Everyone must take a turn sharing teacher and student work.

Procedure:

Facilitator reviews the norms the group has established (5 minutes)

Presenting teacher presents the context of the work (describes the unit in which the assignment was used, including where the task (assignment) fit in the unit.
Facilitator asks participants if there are any clarifying questions, questions which involve only a very brief, factual answer. (10 minutes)

Presenting teacher presents task (assignment) just as it was given to students and participants try to complete the assignment in silence, making brief notes (10 minutes)

Facilitator proceeds with Step 1 of the Placemat. At this time the presenting teacher sits silently reflecting on the discussion of the assignment. It is important for the facilitator to remind everyone that this is not an evaluation of the teacher or the work and the teacher is not there to defend what they have done. It is a time for reflection. (10 minutes)

Presenting teacher shares three sample of student work, one which met expectations, one which exceeded expectations, and one which does not yet meet expectations. All names and no marks should appear on the samples. Participants observe or read the work in silence, making brief notes about whatever they observe in the work.(10 minutes)

Facilitator proceeds with step 2 of the placemat. Again the presenting teacher is silent, reflecting on the conversation taking place. At no time should the student be discussed, only the work. (10 minutes)

Presenting teacher presents their assessment of the work. (5 minutes)

Facilitator proceeds with step 3 of the placemat (5 minutes)

Facilitator proceeds with step 4 of the placemat allowing the **presenting teacher** to respond and share reflections on what they heard and potential changes to the assignment, assessment, and instruction. (10 minutes)

Facilitator invites all participants to share thoughts they have about their own teaching, students' learning, or ways to support student learning. (10 minutes).

For additional information on implementing AERO E2E in your school
Contact Erma Anderson at ermaander@gmail.com

Preventing Students from Becoming Low-Math Achievers

Nell Cobb, DePaul University
William Crombie, Algebra Project

February 2010

The Algebra Project (AP) is a direct descendent of the community organizing tradition of America's Civil Rights Movement. Bob Moses, the founder of the Algebra Project, as field secretary of the Student Non-violent Coordinating Committee in Mississippi during the 1960s, was a community organizer working with poor Black sharecropper communities in the Mississippi Delta. The mission then was to assist members of those communities to gain the right to vote and consequently more fully participate in American society. The mission, today, for the Algebra Project is, similarly, to work with disenfranchised communities of young people in order to assist them in becoming more fully citizens of 21st century America. A necessary condition for full participation in our society requires more than just the reading/writing literacy of 20th century. Young people today need in addition to the literacy standards of the last century a higher standard of mathematical literacy – the ability to read, write, and reason quantitatively.

The Algebra Project has chosen to work with the lowest performing schools and students in underserved urban and rural communities. The traditional approach has been to arrest the development of underperforming students, impede their access to higher studies, and remediate them until they meet a specified performance benchmark. This strategy, a form of cognitive Taylorism, focuses on basic skills, is typically highly procedural, and shows little hope of effectively engaging these populations of students. Over the past 25 years the Algebra Project has developed an approach to working with these populations of students which is both culturally sensitive and experientially-based. Given the present impasse in raising the achievement of the lowest performing student in the US we believe our Cohort Model ultimately offers an effective alternative to current remediation strategies.

The Algebra Project's Cohort Model is a program to accelerate the mathematical learning of students previously under-performing in mathematics. It is based on experiences at Lanier High School in Jackson, MS and at Edison High School in Miami, FL. The Cohort model involves keeping a group of students together for math instruction from grade 9 through grade 12. The students, over time, develop a commitment to their own and each others success in achieving their target of mathematical literacy. Based on our past experience, we have identified features of the cohort model that we find will enable students who enter high school performing in the bottom quartile on national or state tests to become prepared for college study.

Essential Features of the Cohort Model

1. A Cohort school commits for four years to maintaining a class size of 20 students, providing mathematics and English classes every day for a 90-minute block, and a providing a common planning period for teachers everyday; and
2. Cohort students commit to take math and English classes every day for 90 minutes and to

participate in summer institutes as well as other aspects of the program listed below; and

3. In the mathematics program students use the Algebra Project's experientially-based classroom materials for all four years; and
4. The students' teachers are prepared and supported in the use of these materials by 2-6 weeks of summer and winter Professional Development (PD) institutes annually, as well as receiving job-embedded daily support during the school year with a experienced Algebra Project professional development specialist in mathematics and a corresponding professional development specialist in English; and
5. Cohort students attend summer institutes (that are locally developed and designed) to enhance their learning.

The Algebra Project has also identified additional characteristics that we recommend as important for cohort schools, but the forms in which these characteristics are implemented will have to vary from school to school. .

- Community groups develop a local network of parents, school personnel, community activists, and leaders, who focus on sustaining the above intervention; and
- Cohort students have support from counselors who can work with them individually and/or in groups; and
- Cohort students are exposed to the wider culture in some way that overcomes their isolation from the larger society; and
- Cohort students receive some form of support for college entry and an introduction to various careers and job opportunities.

Expected Outcomes for Students

If the five Cohort Characteristics are implemented, we predict the following outcomes for cohort students. These outcomes are the focus of a research effort presently being conducted at four sites participating in a five-year, National Science Foundation, Discovery Research initiative (#DRL0822175):

1. More than 90% of the cohort students will chose to remain in the program for all four years;
2. Cohort students will exhibit-
    a. positive attitudes toward mathematics and confidence in their own mathematical thinking;
    b. a desire and capacity to engage in deep mathematical thinking about various concepts;
    c. a willingness to demand engagement from their peers, and to take responsibility for the classroom environment;
    d. an insistence on support from adults, including teachers, parents, administrators, and government officials.

In addition there are three critical and measurable student outcomes that we are looking to achieve in the Cohort Program:

1. All cohort students who remain will pass state mathematics tests and mathematics sections of graduation exams;
2. All cohort students who remain will perform sufficiently well on college entrance exams (SAT or ACT) to gain admission into college; and
3. A large majority of the cohort students who remain will place out of remedial math in college and will be qualified to enroll in mathematics courses for college credit.

The Process of Program Implementation

The Algebra Project's Cohort Program involves three areas of development and an organizing strategy. In order to implement the Cohort Program the AP has been involved in Materials Development, Professional Development and Professional Development for Professional Developers.

Materials Development

From 2003 to 2008, with the support of two grants from the National Science Foundation's Instructional Materials Development Program, the Algebra Project began the development of materials for high school based in part on lessons previously learned from work with middle schools and middle school curricula during the 1990s. These materials, however, were developed in partnership with university research mathematicians and math educators. The most salient features of the materials were:

- As with the middle school materials, the high school materials were experientially-based. By that we mean that the central concepts of the curricula were introduced by some common experience for the students. In the first instance the purpose of the experience is to help students take ownership over the process of learning by giving them, first and foremost, an emotionally and intellectually engaging context to consider and reflect upon.
- The materials, themselves, are conceptually-challenging. But in adopting an experiential approach to the mathematics they are also leveling the playing field. That is to say that while the entry point they provide into the mathematics does not require specialized knowledge the materials do require careful reflection on the part of students. The materials are built to support and promote sense-making, conjecturing, and reasoning.
- The experiences were specifically chosen and constructed in order to establish a grounding metaphor for students, i.e. a way for students to make sense of abstract mathematical ideas by reference to these core experiences.
- The materials guided students, via a curricular process, in mathematizing their experiences to finally arrive at the abstract symbolic characterizations of mathematical concepts that the high school curriculum requires.
- And finally because the materials built the mathematics in this experiential fashion the curricula provided a concrete basis for student voice and discussion. Consequently the curricular materials put a great deal of emphasis on developing both the orality and literacy of students.

Professional Development

Materials structured in this fashion make qualitatively different and greater demands on teachers than more traditional curricula or even, in some instances, the more recent reform curricula in mathematics. The professional development of teachers in Cohort Programs requires support for a deep change in teacher practice. And by practice, we are referring to both a teachers' practice of teaching (pedagogy) but also their practice of learning – the teachers' relationship to mathematics as a discipline, how they conceptualize, and even more importantly, how they re-conceptualize the mathematics they are to teach. The experiential approach to mathematical concepts and the prominence given to student voice means that teachers find themselves facilitating a much more open discussion of mathematical topics than most mathematics teachers in the US are use to.

We find these demands are best met by helping teachers organize themselves into effective professional learning communities. In the process of working together in small and large collaborative groups teacher practice, both the practice of teaching and the practice of learning, necessarily become public, a public practice open to observation, comment, and critique by peers. Within this context, the professional development of teachers in a Cohort Program is directed towards helping teachers, both individually and collectively, achieve:

- Content Mastery – a deep re-conceptualization of the mathematics they need to teach, its conceptual precursors and consequences.
- Lesson Mastery – the ability to plan, deliver, and assess "polished" lessons.
- Classroom Leadership – providing more than just (classroom) management for the Cohort of students that they teach, support, and develop.

Professional Development for Professional Developers

The Algebra Project's work with high school as opposed to the work with middle schools in the 1990s has highlighted the need for much stronger collaboration between AP professional development specialists, mathematicians and math educators, and the need to focus on increasing both the quality and capacity of professional development services for teachers, PD specialists and mathematicians at all levels.

This need is demonstrated in at least two basic ways. In its efforts to promote math literacy for under-served populations, the Algebra Project developed at the middle school level and is presently developing at the high school level conceptually-challenging curricula delivered through an experientially-based pedagogy. This combination of content and process presents unique demands upon both the teachers who will implement AP curricula and the research mathematicians and AP PD specialists working with and supporting these teachers. As mentioned above, it requires teachers to re-conceptualize the mathematics they have been teaching at a much deeper conceptual level then they previously have. But it also requires research mathematicians and AP PD specialists to work closely together in order to aid and support teachers in this process.

The nature and type of work at the high school level requires an especially deep collaboration between teachers, research mathematicians and AP PD specialists. The nature of this work places the culture of research mathematicians' right next to and more often within the culture of the k-12 educational system. These two cultures often carry very different norms and expectations. Consequently this work requires from both research mathematicians and AP PD specialists not only a high degree of cultural sensitivity but also an ability to effectively negotiate the cultural differences between these groups.

The Algebra Project's PDPD Program is consequently aimed at supporting the development of professional development specialists and university mathematicians who are working with cohort schools to provide on-going support to Algebra Project teachers. This begins with the delivery of a 5-day *Professional Development for Professional Developers Institute* appropriate for AP PD specialists and AP mathematicians at both the secondary and middle grades levels. It continues periodic support for those supporting cohort teachers.

An essential tool which the Algebra Project is the development of a competency model of effective PD practice, a Model of Excellence, for AP mathematicians, AP PD specialists, and the teaching of teachers within the Algebra Project at the secondary school level.

Organizing Students – Building Demand for Math Literacy

Founded in 1996 as an outgrowth of the Algebra Project**,** it is the mission of the Young People's Project (YPP) to use Math Literacy as a tool to develop young leaders and organizers who radically change the quality of education and life in their communities so that all children have the opportunity to reach their full human potential. YPP trains and organizes high school and college students to work with their younger peers and families, and in their communities, to build demand for math literacy. Math literacy is the point of entry to, and foundation for, the broader leadership development and skills-building YPP engages in to develop young people as effective organizers and advocates for high quality public education in their communities. YPP operates from the premise that there cannot be successful school reform without community reform, i.e. the culture of the community around education has to change. This cultural transformation must take place on three levels: 1) communities develop confidence that their youth are capable of academic success; 2) communities develop a sense of responsibility for ensuring the academic success of their youth; and 3) communities begin to see themselves as agents of social change, responsible for building the requisite demand and capacity necessary to ensure high quality public education for all youth. The Young People's Project is thus the organizing strategy for developing student leadership within the Cohort.

The Cohort Model in Mississippi

The Cohort Model was initiated at Lanier High School, Jackson, MS. Beginning with those students who took Algebra I with the Algebra Project in 2002-03, the project kept together a group of students, who took math every day in long periods. The graphs below show the results at Lanier High School, where the first cohort graduated in 2006. The features of the model are based on work at Lanier and are the result of collaboration among teachers, students, Algebra Project members, and university and math educators.

Through a grant from the National Science Foundation's DRK-12 program the Algebra Project has established six 9th grade cohorts in four schools: Crenshaw and Franklin High Schools in Los Angeles, CA; Mansfield High in Mansfield, OH; Eldorado High in Eldorado, IL; and Ypsilanti

High in Ypsilanti, MI. With this grant, the Algebra Project holds itself accountable to radically transform the lives of additional students who have so far not been reached by education reforms, and to stimulate the interest of educators across the economy in this model.

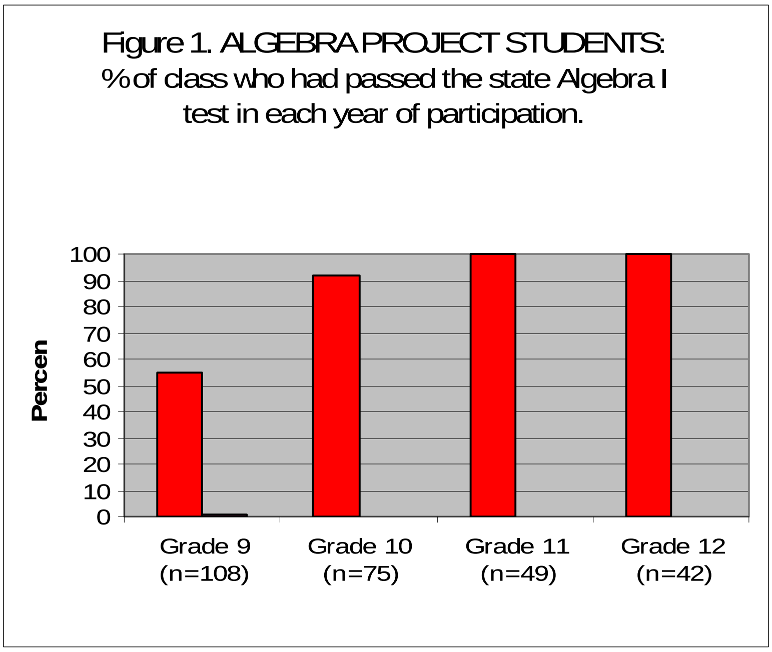
Results from the first Cohort in Jackson MS
The students, who took Algebra I with project-trained teachers in 2002-03, were offered the chance to stay together for the next year, taking math daily in double periods. This was repeated each year. We later tracked all of the students at Lanier who took Algebra I in 2002-03, and compared outcomes for the Algebra Project students (108 in Grade 9) and those who took Algebra I using traditional materials (86 in grade 9).

The Algebra Project students had improved outcomes in passing the state Algebra I test, in on-time graduation rates, and fewer students dropped out of school, compared to the non-Algebra Project students.

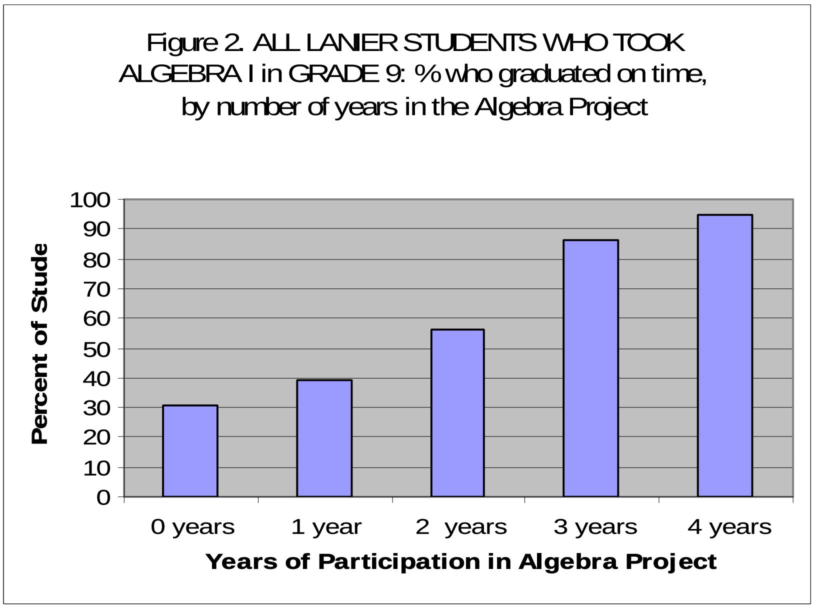
These effects strengthened after TWO years in the project (see tables below). This is due to the project's strategy to enable students who failed the state test at the end of Grade 9 to continue with the project's Geometry course in Grade 10, with extra support to re-take the state test. The project's approach places value on student outcomes, in contrast with the school accountability approach embedded in the NCLB policies. Results are also attributable to the formation of the peer culture for persistence and achievement, enhanced by keeping students together.

1. **Increase in the proportion of students who passed the state Algebra I test (required for graduation).** At the end of 2003, all Algebra I students took a state Algebra I test. Fifty five percent of Algebra Project students passed compared to 37% of non-Algebra Project students. After two years, 92% of Algebra Project students had passed.
2. **Increase in the on-time graduation rate.** 53% of students in the Algebra Project who stayed in the cohort for at least one year graduated on time, compared with 31% of students never in the project; however, if students remained in the project for at least two years 69% graduated on time, compared to 27% who were in for only one year, or not at all.
3. **Decrease in the proportion of students who drop out of high school**. Only 17% of students who were known to have dropped out of high school, compared to 35% for those never in the project (based on state records).

Algebra Project Cohort I (2002 – 2006)
Lanier High School, Jackson, MS



Nearly all students passed the state Algebra I exam by the end of sophomore year



On-time graduation increased with time in the Algebra Project Cohort

Adoption to APEC economies

In order to develop and implement a Cohort Program APEC economies would need to consider experiential learning, culturally relevant curriculum built upon a cultural base of literacy. The curriculum would focus on the language and experiences of their students just as the AP curriculum in the US focuses on the respective student populations.

The training requirements could best be met through the establishment of a Design Team. Such a team would need to consist of community organizers, university mathematicians and math educators, school administrators, professional developers, and teachers. We recommend that this Design Team complete an apprenticeship with an Algebra Project Cohort site. The apprenticeship would be the first step in a seven year implementation strategy.

Challenges

The principal challenge for an APEC economy would be the time and financial commitments for the projected seven year term. Short term commitments are easier than long term commitment, at least in the short term. We have outlined a seven year proposal for program adoption.

A Seven Year Implementation of a Cohort Program

- Year One – Design Team apprenticeship with the Algebra Project and an Algebra Project Cohort site.
- Year Two – Materials development in classrooms involving mathematicians, teachers and other design team members.
- Year Three to Six – Cohort development across four years
- Years Five and Six – Development of a Model of Excellence for Teachers and Professional Developers
- Year Seven Summative evaluation of the Cohort Program

Coda

Human resources, unlike natural resources, do not sit still waiting to be developed. Human resources are always in motion. If they are not progressing they are regressing. Young people, students, from the most under-served, disenfranchised parts of our societies are either tomorrow's source for innovation and discovery or tomorrow's source for social instability. The Algebra Project chooses to work with those students who society has seemingly given up on. The Algebra Project is looking for a path to accelerate their development rather than remediate. And most importantly the Algebra Project locates the key to success with those young people themselves, with the demand they raise for a quality education, with the demands they make of themselves.

References

Algebra Project TOT Program, (1998). *The Algebra Project Report on the Model of Excellence*. Cambridge, MA: Algebra Project.

Dononvan, M. S., and Bradford, J. D. eds. (2005). *How Students Learn: History, Mathematics, and Science in the Classroom.* Washington, DC.: National Academy Press

Kolb, D. (1984) *Experiential Learning: Experience as the Source of Learning and Development*. Englewood, NJ: Prentice-Hall.

Moses, R. P. (2009). An Earned Insurgency: Quality Education as a Constitutional Right, *Harvard Educational Review*, 79(2), 370-381.

Moses, R. P. & Cobb, C. E. (2001). *Radical equations: Math literacy and civil rights*. Boston, MA: Beacon Press.

Payne, C. M. (1995). *I've Got the Light of Freedom: The Organizing Tradition and the Mississippi Freedom Struggle*. Berkeley: University of California Press.

Spencer, L. M. and Spencer, S. M. (1993) *Competence at Work: Models of Superior Performance*. New York: John Wiley and Sons.

Taylor, F. W. (1998). *The Principles of Scientific Management.* Mineola, NY: Dover Publications.

A Combined Abduction-Induction Strategy in Teaching Mathematics to Gifted Students-with-Computers through Dynamic Representation

Assoc. Prof. Dr. Tran Vui, Hue University

February 2010

1. Introduction

Students today grow up in a culture, which mostly depends on visual representations, messages are delivered dynamically through pictures. Students are used to receiving information in a very active mode. While written forms of representation are still important, it is necessary to consider how mathematical ideas can be represented through a visually dynamic medium. This strategy itself may help some gifted students to investigate and explore interesting mathematical ideas in a new and different way.

Unfortunately, in most current enriched or accelerated programs for mathematically gifted, students often just solve more problems at faster rate without opportunities to develop their own mathematical ideas. As a result, gifted students rely mostly on procedural knowledge. They lack the opportunities to engage in challenging investigations, experimental environments, and higher-level mathematical thinking.

Gifted students in most schools now have access to computers in their classrooms, and an increasingly large percentage of these students have private computers at home. As the goals for technology education and the promises of educational change have grown, the hardware and the software used in both schools and homes have improved steadily (Holden, 1989). Students are provided opportunities to do research and apply complex thinking skills by working with real problems and computer simulations. Learning becomes fun and more challenging. Students are taught programming languages that aid them in making a computer become a real tool. All students in gifted and talented programs should be introduced to such computer applications and programming.

The use of multiple dynamic representations which promote students' exploration of mathematical ideas is relevant. Research indicates that positive gains in understanding of mathematical topics appear in cases when multiple modes of dynamic mathematical representations are used effectively. Multiple modes of representation improve transitions from concrete manipulation to abstract thinking, and provide a foundation for continued learning. This study investigates the effectiveness of experimental environments for gifted students-with-computers to explore mathematical ideas through dynamic multiple representations. The purpose of this talk is to share an combined abduction-induction strategy in teaching mathematics to gifted in experimental environments. Applying this strategy, students need to construct their own dynamic models to conduct their experimentation.

2. Dynamic Visual Representations

This section emphasizes some of the positive effects of visualizing in mathematical concept formation and to show how dynamic visual representations can be used to achieve more than just a basic, procedural and mechanical understanding of mathematical concepts.

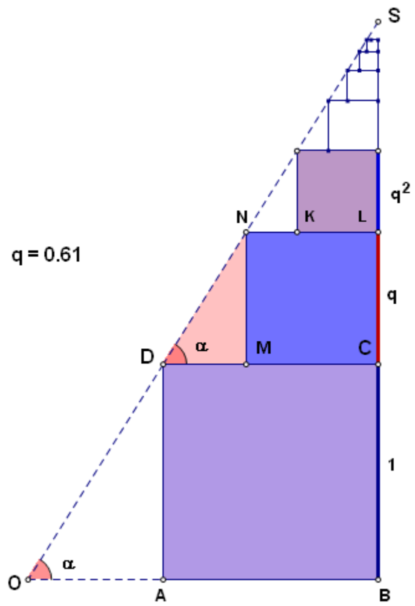
Arcavi (2003) proposed that:

> *Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings.*

Example. Given a square *ABCD* with the side of 1 unit, on segment *CD* take a moving point *M*. Let $CM = q$; $0 \le q \le 1$. We assume $q \ne 0,1$. Construct square *MCLN* with the side *q*.

And then we continue to construct square with the side $KL = q^2$ and so on. Use a dynamic software to construct a geometric model for this situation and then find the sum of the infinite geometric series:

$$BS = 1 + q + q^2 + \cdots + q^n + \cdots$$



Investigation. From the picture, $DM = 1 - q$, so in the right triangle *DMN*, we have:

$$\tan\alpha = \frac{MN}{DM} = \frac{q}{1-q}$$

In right triangle *OAD*,

$$OA = \frac{1}{\tan\alpha} = \frac{1-q}{q}$$

We have: $OB = OA + AB = \dfrac{1-q}{q} + 1 = \dfrac{1}{q}$. In right triangle *OBS*,

$$BS = OB \times \tan\alpha = \frac{1}{q} \times \frac{q}{1-q} = \frac{1}{1-q}$$

In the dynamic models contructed by students, they can drag moving point *M* to change the value of *q* and observe the behaviour of the sum *BS*.

The computer is a rich source of visual and computational images that makes the exploration of mathematical conjectures possible. In this sense, the function of the software is paramount, providing the students with the opportunity to explore mathematical ideas, analyze examples and counter-examples, and then gain the necessary visual intuitions to attain powerful formal insights. However, it seems that, although visualization is recognized as relevant, the final objective continue to be the rigorous mathematical proof, as we already reviewed within communities of mathematicians.

A *visual approach* in the mathematical thinking process would be characterized by:

- Use of graphical information to solve mathematical questions that could also be approached algebraically.
- Difficulty in establishing algebraic interpretations of graphical solutions.

- No need to first run through the algebra, when graphical solutions are requested.
- Facility in formulating conjectures and refutations or giving explanations using graphical information.

In this case, the computer is used to verify conjectures, to calculate, and to decide questions that have visual information as a starting point.

3. Dynamic Multiple Representations

The use of multiple mathematical representations has been shown to increase students' capability in exploring of mathematical ideas. Nonetheless, while research indicates positive gains in student learning of mathematical topics, these gains appear in case when the multiple modes of mathematical representations are used effectively. Multiple representational software has been developed for computers at a speed which is difficult to keep up with. The importance of such an approach is it facilitates students' coordination of established mathematical representations such as tables, Cartesian graphs and algebraic expressions. Dynamic multiple representations could actually change the way students and teachers know about mathematics. We believe that computer technology and its different interfaces are changing the nature of the senses we use to communicate within a gifted student-with-computer.

Example: Dynamic multiple representations of fractions, percentages designed by The Geometer's Sketchpad. This model represents the relationship among fractions, percentages and bar chart at the same time on the screen with the values automatically changed when we drag *a*, *b*, or *c*.

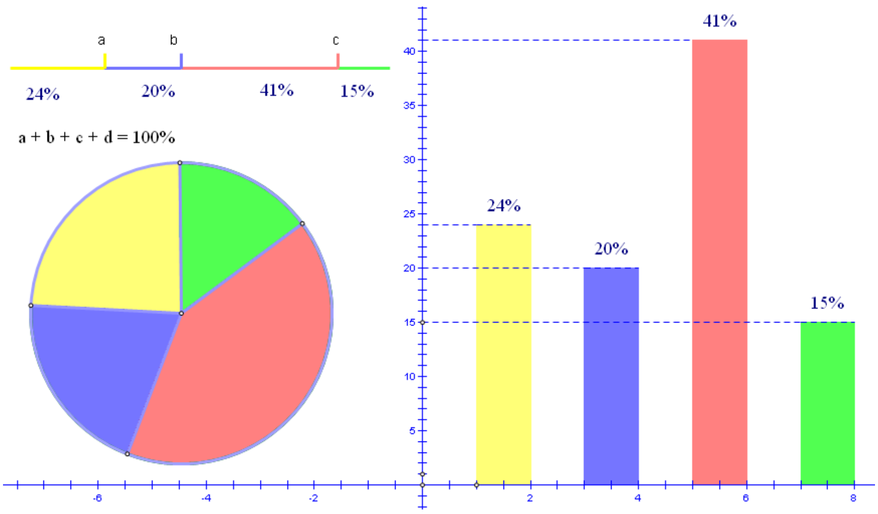


Figure 1. *Dynamic multiple representations of fractions, percentages and bar charts*

Manipulatives

The replication of physical manipulatives in the form of computer applications provides additional features and advantages over traditional manipulatives. Virtual manipulatives are advantageous in their capability to connect dynamic visual images with abstract symbols where physical manipulatives have limitations. Unlike physical manipulatives, virtual manipulatives use graphics, numbers, and words on the computer screen to connect the iconic with the symbolic mode and virtual manipulatives can record user interaction with the virtual manipulative and record such as movements and screen capture across time so the student or teacher can understand the false starts as well as the final submitted solution.

4. Gifted Students-with-Computers Exploring Mathematical Ideas in Experimental Environments

Experimentation associated with computers has a paramount role in mathematics education. Experimental mathematics has gained respectability in recent years, and that computers are partly responsible for this change. Mathematicians carry out experimental mathematics before the formulation of a conjecture they believe to be true and before the construction of a logic proof. Experimentation should be present more in schools for gifted students because computers are more available there, and experimentation is in resonance with collectives which involve computer technology. The use of geometrical software such as LOGO, Cabri or Sketchpad, the computer algebraic systems such as Maple, Derive, Mathematica, the graphing calculator or the so-called microworlds generate experimental environments that can be considered as laboratories where mathematical experiments are performed.

The experimental-with-computer approach

Educated trial and error, conjectures and refutations were elements of the logic of mathematical discovery. These elements characterized the students learning process in an experimental approach. More recently, we have started to think about this process as a way of thinking which is neither deduction nor induction but abduction, since the trials are very quickly no longer random. The logic of discovery is another way of characterizing abduction in the classical sense as described by C. S. Peirce. Abductive reasoning entails the study of facts and the search for a theory to explain them. It is the mode of inference dealing with potentiality:

- possible resemblance;
- possible evidence;
- possible rules leading to plausible explanations;
- possible diagnostic judgments;
- clues of some more general phenomenon.

Abduction

Reasoning which produces prediction and reasoning which explains observations are here considered cognitively different. The first is closely related to classical logical deduction, and the second is closely related to abductive reasoning and creative thinking. If Pierce uses abduction and deduction, Polya analogously introduces the idea of plausible reasoning in contrast to demonstrative reasoning (Polya, 1968). The relationship among abduction, induction and deduction was illustrated in Figure 2 of taxonomy of the inferential trivium.

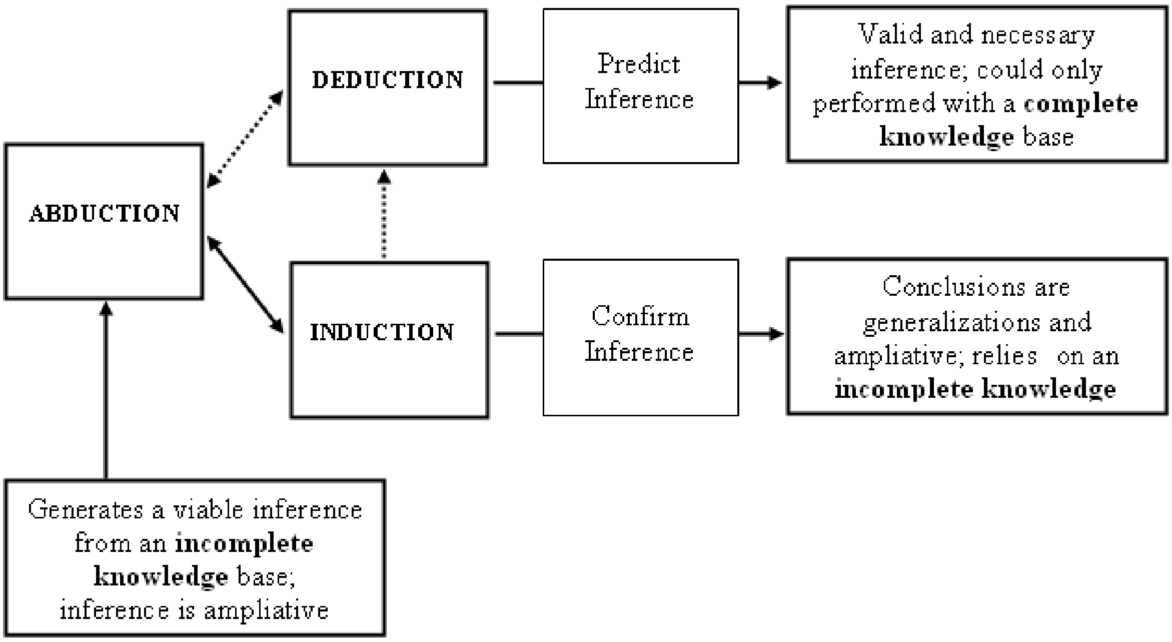


Figure 2. *Taxonomy of the inferential trivium* (Revised from Rivera and Becker, 2007).

For Abe (2003), Peircean abduction is another form of discovery or suggestive reasoning that "discovers new events" (p. 234) and yields explanations rather than predictions because they are not directly knowable. It is similar to induction insofar as both are concerned with discovery. However, it is distinguished from induction in that the latter "discovers tendencies that are not new events" (p. 234). Induction tests an abduced hypothesis through extensive experimentation and increased success on trials means increased confidence in the hypothesis.

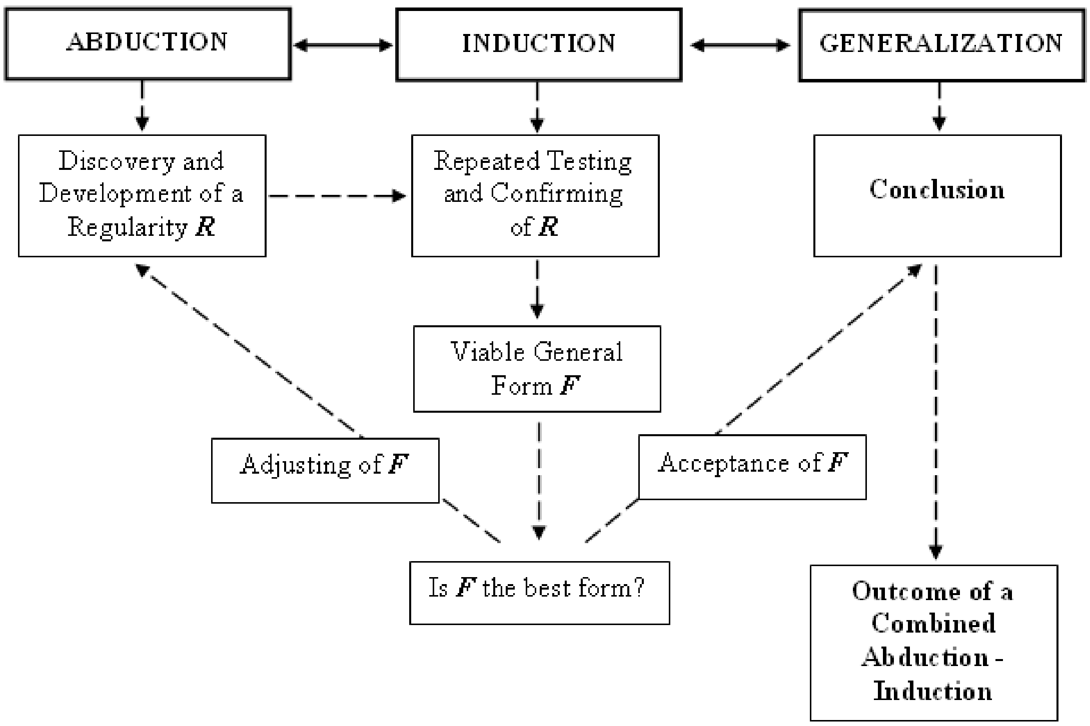


Figure 3. *Pattern Generalization Scheme* (Revised from Rivera and Becker, 2007).

Figure 3 illustrates how the combined process materializes in a generalization activity from the beginning phase of noticing a regularity R in a few specific cases to the establishment of a general form F as a result of confirming it in several extensions of the pattern and then finally to the statement of a generalization (Rivera and Becker, 2007).

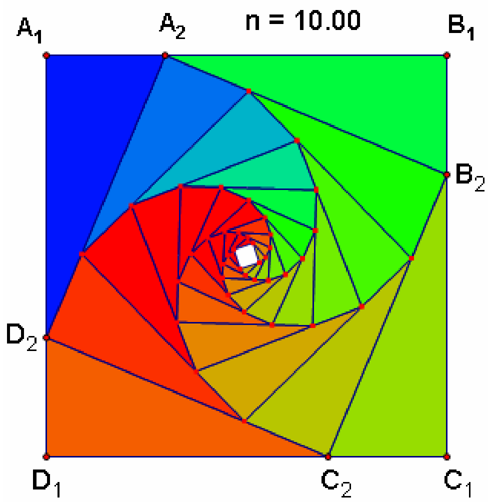
Example: Rotation, Dilation and Iteration

Given a square $A_1B_1C_1D_1$ with the side of 4 units. Construct square $A_2B_2C_2D_2$ as follows:

- Take A_2 arbitrary on A_1B_1. Calculate the ratio $\frac{A_1A_2}{A_1B_1} = k$. Then $0 < k < 1$.
- Construct B_2 that satisfies $B_1B_2 = A_1A_2$. Two points C_2 and D_2 are constructed similarly.
- From the square $A_2B_2C_2D_2$, construct square $A_3B_3C_3D_3$ as above and so on... $A_nB_nC_nD_n$,...

Let u_n be the side of the square $A_nB_nC_nD_n$. With a dynamic geometric software such as The Geometer's Sketchpad, students can construct their own model as shown in the below figure (Tran Vui, 2007b).

Students drag moving point A_2 to observe the change of the figure, there are many mathematical facts in this figure such as: rotation, dilation, iteration, geometric sequence, sum of infinite series… Students need to search for some "new theories" to explain these observed facts. Gifted students can conduct some experiments as follows:



- When $k = \frac{1}{4}$, find the formula of u_n.
- When k is arbitrary, find the formula of u_n in terms of k.
- When k is arbitrary, calculate the sum of the first n terms of the series.
- Define the ration and the dilation in this interation.

Visual representations in mathematics are more concrete and simpler than meanings mediated by verbal language but they are often also clearer and easier to understand. Due mainly to advances in computer software, pictures are today becoming a convenient vehicle for communicating ideas.

We can say that the experimental approach gains more power with the use of computers and thus, the experimental-with-computer approach provides:

- the possibility of testing a conjecture using a great number of examples and the chance of repeating the experiments, due to quick feedback given by computers;
- the chance of getting different types of representations of a given situation more easily;

- a way of learning mathematics that is resonant with modeling as a pedagogical approach.

5. A Combined Abduction-Induction Strategy in Teaching Mathematics to Gifted Students-with-Computers

The visual nature of multiple dynamic representations allows students to investigate algebraic and geometric properties. The dynamic visual representations on the screen generate experimental environments for students to manipulate with mathematical objects. Students have more opportunities to observe, test their guesses, make conjectures or counter-examples.

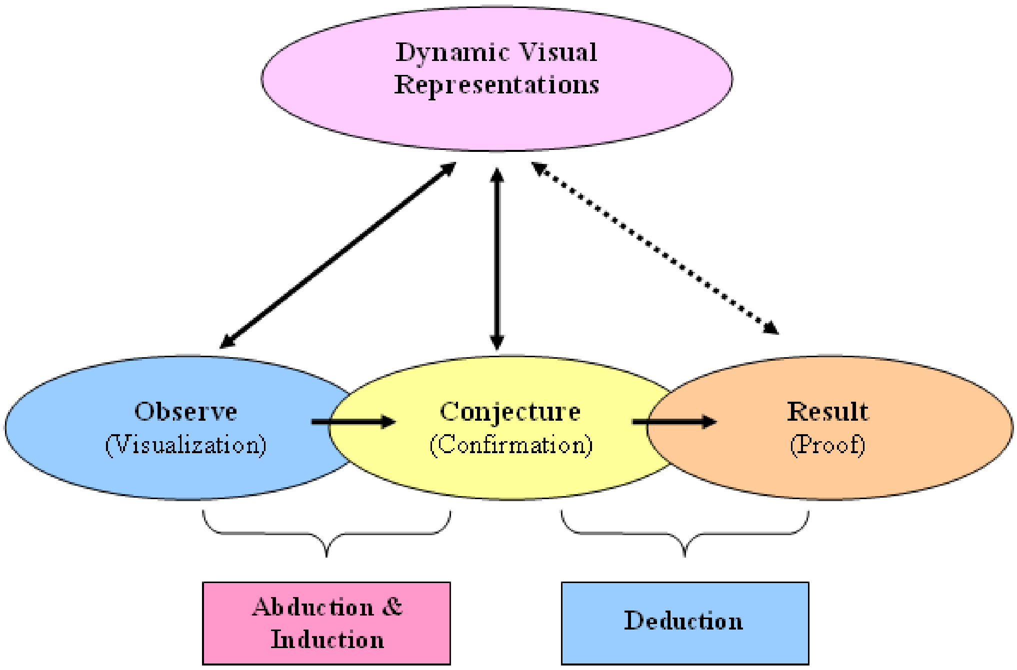


Figure 4. *A combined abductive- inductive strategy in teaching mathematics with dynamic visual representations*

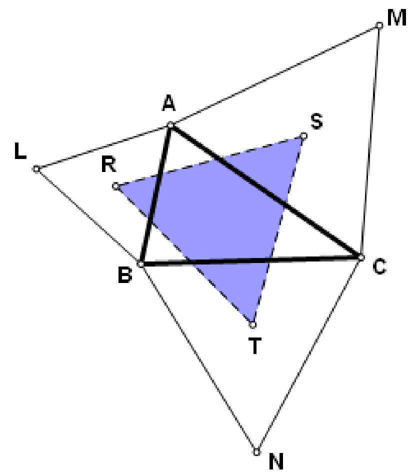
The interplay among dynamic representations, visualization, confirmation and proof can be illustrated by the model in Figure 4.

An investigation with The Geometer's Sketchpad

Example. *Take any generic triangle, and construct equilateral triangles on each side whose side lengths are the same as the length of each side of the original triangle. Surprise*: *the centers of the equilateral triangles form an equilateral triangle*!

1. Construct a triangle *ABC*, any triangle.
2. Construct equilateral triangles on the sides of the triangles.
3. Construct the centers of these triangles.
4. Connect the centers of these triangles. What is true?

Drag *A*, *B* or *C* to observe and collect the lengths of three sides *RS*, *ST*, and *TS*.



Use the "*tabulate*" option in "Measure Menu" to make a table of data. From these data students can predict a conjecture.

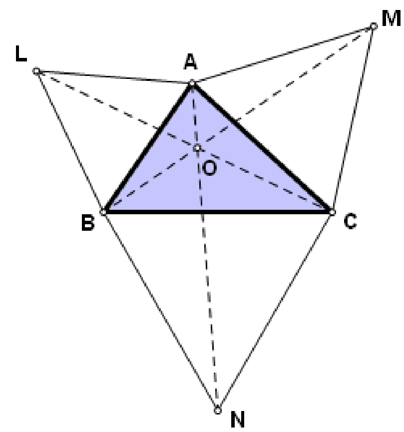
Conjecture 1. *The triangle RST is an equilateral triangle.*

Construct segments *AN*, *BM*, and *CL*.

From the figure as shown on the right, some conjectures can be made:

Conjecture 2. *Three segments AN, BM, and CL are equal.*

Conjecture 3. *Three segments LC, MB and NA intersect at a single point O.*

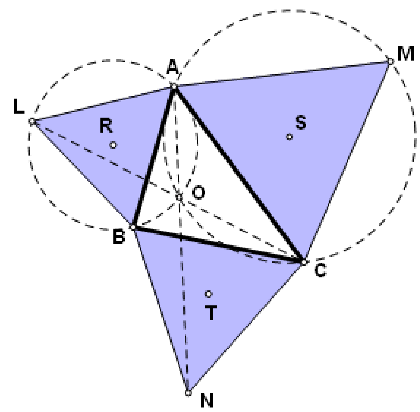


Conjecture 4. $\dfrac{Perimeter\ of\ (\Delta ABC)}{OR+OS+OT}=\sqrt{3}$

Conjecture 5. *The circumcircles of the three equilateral triangles ABL, BCN, and ACM have a common point.*

Conjecture 6. $\dfrac{OA+OB+OC}{ON+OM+OL}=\dfrac{1}{2}$.

Conjecture 7. *Three angles AOB, BOC, and AOC are equal to* 120 *degrees.*

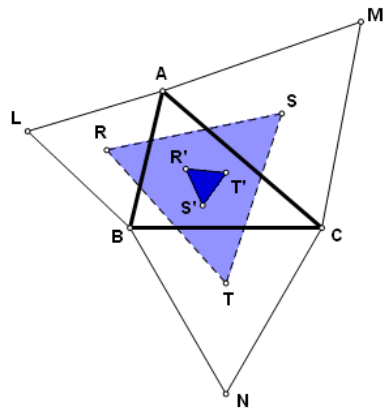


Conjecture 8. *Point O is the unique interior point to the triangle ABC that has the minimum total distance to three vertices A, B, and C.*

Let *R*', S' and *T*' be the image of *R* under the reflections about *AB*, *AC* and *BC* respectively.

Conjecture 9. *The* (*inner Napoleonic*) *triangle R'S'T' is an equilateral.*

Conjecture 10. *The difference in areas between the outer and inner Napoleonic triangles is the area of the original triangle ABC.*



A good dynamic geometric software gives gifted students more opportunities to construct their own models and observe many mathematical facts. Students can use "*Measure Menu*" to measure length, perimeter, angle, area, arc angle, arc length... to get more numerical data. From these data students make more conjectures based on their incomplete knowledge.

6. Training Global Teachers of the Gifted

There is a great need to provide opportunities for the education of teachers of gifted students that is applicable and accountable internationally. Using dynamic softwares to design effective dynamic multiple representations for exploring mathematical ideas require the learners a strong background in mathematics and also their skills in using the dynamic softwares. To prepare leading teachers of gifted know how to use dynamic representations we should:

- build up a knowledge bank of visual dynamic multiple representations or models for main mathematical ideas or challenging real-life investigations.
- set up virtual schools for mathematically gifted focused on dynamic multiple representations available for all gifted and accessible to international students.
- create more chances for teachers and students to explore mathematical ideas through online gifted education with an international and multicultural perspective.
- provide opportunities for teachers of the gifted to work internationally, and become global leading teachers themselves, are considerable in APEC region with the support of the internet.

The internet allows immediate and flexible access to vast resources, materials and leading mathematics teachers, and has changed the concept of knowledge from stable forms to fluid and fast changing. New roles for the teacher of the gifted beyond the classroom, faciliators encompass the monitoring, management and creative use of online formats in virtual environments accessible at any time, anywhere.

7. Conclusion

Do not force mathematically gifted students learn too much the knowldege invented by mathematicians long time ago. We should generate good educational environments for students to generate a viable inference from their incomplete knowledge base. Experimental environments based on dynamic multiple representations encourage gifted students to incorporate many different types of representations into their sense-making, the students will become more capable of solving mathematical problems and exploring underlying mathematical ideas. Dynamic mathematical softwares generate environments that can be considered as laboratories where mathematical experiments are performed. Trial and error, conjectures, refutations and generalizations are elements that characterize gifted students' work in these experiemental environments.

References

Abe, A. (2003). Abduction and analogy in chance discovery. In Y. Ohsawa & P. McBurney (Eds.), *Chance Discovery* (pp. 231-248). New York: Springer.

Arcavi, A., 2003, *The role of visual representations in the learning of mathematics*, Educational Studies in Mathematics 52: pp.215–241, Kluwer Academic Publishers. Printed in the Netherlands.

Eisner, E. W. (1997). *The enlightened eye: Qualitative inquiry and the enhancement of educational practice* (2nd ed.). New York: Merrill Publishing Company.

Holden, C. (May 26, 1989). Computers make slow progress in class. *Science*, pp. 906-909.

Marcelo C. Borba and Mónica E. Villarreal (2005), *Humans-with-Media and the Reorganization of mathematical Thinking: Information and Communication Technologies, Modeling, Visualization and Experience*, Mathematics Education Library, Vol. 39, Springer Science & Business Media, Inc., USA..

Rivera, F. & Becker, J. (2007). *Abduction in pattern generalization*. In W. Jeong-Ho, L. Hee-Chan, P. Kyo-Sik, & J. Dong-Yeop (Eds.), Proceedings of the 31st conference of the International Group for the Psychology of Mathematics Education, Vol (4), pp. 97–104. Seoul, Korea: Korea Society of Educational Studies in Mathematics.

Tadao Nakahara, 2007, *Development of Mathematical Thinking through Representation: Utilizing Representational Systems*. Progress report of the APEC project "Collaborative studies on Innovations for teaching and Learning Mathematics in Different Cultures (II) - Lesson Study focusing on Mathematical Communication". Specialist Session, December 2007, University of Tsukuba, Japan.

Tran Vui et al. (2007a). *Design dynamic models for teaching and learning high school mathematics with the Geometer's Sketchpad*, Educational Publishing House, MOET Viet Nam.

Tran Vui et al. (2007b), *Exploring Algebra and Analysis* 11 *with the Geometer's Sketchpad,* Educational Publishing House, MOET Viet Nam.

Tran Vui et al. (2007c), *Exploring Geometry* 11 *with the Geometer's Sketchpad*, Educational Publishing House, MOET Viet Nam.

Tran Vui (2009). *Use mathematization to enhance mathematical literature of high school students.* Jounal of Educational Science, NIESAC Viet Nam, ISSN 0868 – 3662, No. 43: 4/2009, pp. 23-26.

Tran Vui et al. (2009), *Design dynamic models for teaching and learning middle school mathematics with the Geometer's Sketchpad*, Educational Publishing House, MOET Viet Nam.

Taking Off With Numeracy: Helping students to catch up

Peter Gould, New South Wales Department of Education and Training, Australia

February 2010

Taking Off With Numeracy (TOWN) is the current title of a program designed to address the persistence of highly inefficient methods of calculating. Imagine a 12-year-old student being told that there are 56 people coming to a party and that each table at the party can seat 8 people. If the student is then asked how many tables are needed for the party, you might not be surprised that the student reaches for pencil and paper to work it out. However, if the student then starts to make marks on the paper for each person, counts and circles each group of eight before finally counting the number of groups, you have seen an inefficient method of counting by one used to solve a division problem.

Figure 1. Using inefficient methods for division

Count-by-one strategies are a normal part of the development of children's mathematical knowledge. Over time children usually develop a range of methods other than counting by one (Carpenter & Moser, 1982; Fuson, 1988; Steffe, Cobb, & von Glaserfeld, 1988; Wright, 1991; Wright, Martland, Stafford, & Stanger, 2002). For some students the pathway to more advanced methods of working with numbers is passed by and they persist with count-by-one methods (Gray, 1991). In a study of mixed ability children aged 7 to 12, Gray referred to the dominance of strategies of counting by ones in use by less able students, and concluded that… *in one sense they make things more difficult for themselves and as a consequence become less able* (1991, p. 570). Gray's comments emphasise the dangers of persistent counting by ones and why these students are on the wrong path—they are on a road to nowhere!

One of the difficulties with inefficient strategies is that, although they are often much slower, they still work. This means that inefficient strategies persist for some children long past the time that they are useful. Indeed, some children simply get faster at using the inefficient method of counting by one. While these methods will often result in the correct answer, they take so much effort that there is little chance of learning new material. The learning of these students in mathematics has reached a plateau.

Taking Off With Numeracy operates as both a whole class program and a within class intervention. The first phase of implementing the program is the identification phase. This involves determining students' current understandings of multi-unit (conceptual) place value. Students' responses to a short assessment are analysed and categorised in relation to what they reveal about students' thinking using a *Place value framework*. The same assessment is used to determine the target group for the in-class intervention. The students in the intervention group are then given a further individual assessment to determine if their problems also relate to specific reading difficulties.

Two assessment schedules are used in the TOWN whole class screening process, one for students in Years 3 and 4, and the other for students in Years 5 and 6. It takes students approximately 15 minutes to complete the initial assessment. To get the most information out of the assessment, the students should be encouraged to show their thinking for solving each task on the sheet. The analysis of the assessment does not focus on the number of correct answers but rather looks at the methods students are using.

The addition of an in-class intervention is provided for classes with significant numbers of students who make extensive use of count-by-one methods. Once the targeted intervention group has been tentatively identified based on the results of the whole-class assessment, a short follow-up individual assessment is administered. The follow-up assessment may also be used with any students where the teacher feels the initial results are not definitive. That is, the follow-up assessment process may also be used for those students whom the teacher was unable to allocate to or exclude from the target group, or for any student in the class about whom the teacher deems it would be useful to find out further information. During the follow-up assessment, students are asked to explain their strategies for solving problems.

The process also involves the use of *Newman's Error Analysis* to identify the point at which students are experiencing difficulty in solving a *word problem*. Through this process, teachers will be able to determine if the students are experiencing difficulty with reading the question, comprehending the question or with the numeracy involved in the question.

Why focus on place value?

Students often develop a simple structural approach to place value. That is, when a student says the "4" in "48" means "four tens", she or he may be demonstrating only verbal knowledge based on the left and right positional labels associated with the digits. Using this approach, a student may not recognise that the "4" represents 40 objects . This distinction between verbal syntactical knowledge and conceptual (semantic) understanding of place value has implications for developing number sense and teaching algorithms based on place value.

The following example of “trading” shows the dominance of routine over understanding.

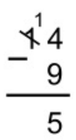


Figure 2. When trading doesn’t help

The student might explain this as, "Nine from four you cannot do, so I trade one ten, nine from fourteen is five". That is, the explanation of trading does not help to solve the problem as it results in additional work with no gain as the final question remains ‘nine from fourteen’. This calculational explanation is different from a conceptual explanation and it is possible that the student is only restating what he or she has heard.

Similarly, a student who does not have a well-developed sense of place value might remember the procedure for addition depends on lining up numbers. To answer 27 + 8, the student writes:

$$\begin{array}{r} 27 \\ \underline{+\ 8} \end{array}$$

Figure 3. Creating the appearance of an algorithm

The procedure followed is then to count on the 8, incrementing the seven by ones to reach 15, writing down the 5 and *carrying* the one. The process of incrementing by ones is usually accompanied by making dots or some form of tally system. This procedure of lining numbers up in columns often passes for an understanding of *place value*.

Rather than relying solely on counting by ones, students need to be assisted to use collection-based methods. Using collection-based procedures can make use of:

doubles, 6 + 6 = 12,

number facts, 6 + 3 = 9,

and tens within place value to "bridge to ten", 19 + 7 = (19 + 1) + 6.

The teaching of algorithms and the use of mental strategies for the four operations with numbers, all rely on an understanding of place value. When this understanding is weak, students use partially remembered procedures or else revert to methods of counting by ones.

Place value understanding also has a significant impact on the use of decimals and related work in measurement. For example, the persistence of unitary counting procedures can result in students being restricted to counting squares to find area or counting individual cubes to determine volume. They cannot successfully use the groupings necessary to determine area or volume efficiently.

Multi-unit place value

Multi-unit place value is evident when students can flexibly regroup hundreds, tens and ones. The student’s concept of ten is central to the development of base ten place value. When students first construct number, they construct a system of ones. Therefore, for most students in Kindergarten and Year 1 in Australia, 24 is 24 ones. At the pre-place value level, both ten and one are treated as simple number words. Although the student can recite the sequence of multiples of ten; “Ten, twenty, thirty, forty…”, these multiples are simply counting numbers in the same way that the words ‘seven’ or ‘thirty-three’ can be thought of as counting numbers.

Reciting the decades does not by itself reflect a sense of increasing the size of the total by ten, that is, incrementing by ten. Indeed, before students have constructed real units of ten and can ***simultaneously*** think about tens and ones, they frequently have difficulty in shifting between units. When presented two lots of 10 and four 1s some students will count by tens saying, '10, 20, *30, 40, 50, 60'*.

The place value framework has been developed on the assumption that to be considered to be on the framework (i.e. at the pre-place value level), a student must be able to count on and back. Typically, students who are pre-place value reconstruct units of ten by counting them. That is, to add twenty they would typically count-on by ones. A student who is not able to count on and back is not on the place value framework.

In recognising that multi-unit place value concepts are difficult for students, we need to differentiate between the difficulty that students have in dealing with representing numbers and the difficulty they have in coordinating abstract units of ones, tens and hundreds (Chandler & Kamii, 2009). The abstraction of the number 24 as a quantity (24 single units, 2 units of ten and 4 ones or even 1 unit of twenty and 4 ones) rather than the number word 'twenty-four' is also different from numeral representations '24' or XXIV. The errors students make in using numeral representations make it clear that the link between the numerals and the abstract quantities they represent is often missing (Figure 4).

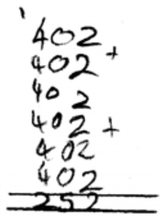


Figure 4. Representing six lots of 402 as 252

Students who predominantly use count-by-one methods with addition and subtraction tend to carry the process over to multiplication and division. The basics of multiplication and division are taught up to the end of Year 4 in schools in New South Wales. Consequently, interventions addressing multiplication and division are a component of the implementation of TOWN in Years 5–6, but not in Years 3–4.

Newman's error analysis

As well as developing a conceptual understanding of place value underpinning operating with numbers, students need to comprehend problem contexts. The Australian educator Anne Newman (1977) suggested five significant prompts to help determine where errors may occur in students' attempts to solve written problems. She asked students the following questions as they attempted problems.

1. *Please read the question to me. If you don't know a word, leave it out.*

2. *Tell me what the question is asking you to do.*

3. *Tell me how you are going to find the answer.*

4. *Show me what to do to get the answer. "Talk aloud" as you do it, so that I can understand how you are thinking.*

5. *Now, write down your answer to the question*.

These five questions can be used to determine why students make mistakes with written mathematics questions.

A student wishing to solve a written mathematics problem typically has to work through five basic steps:

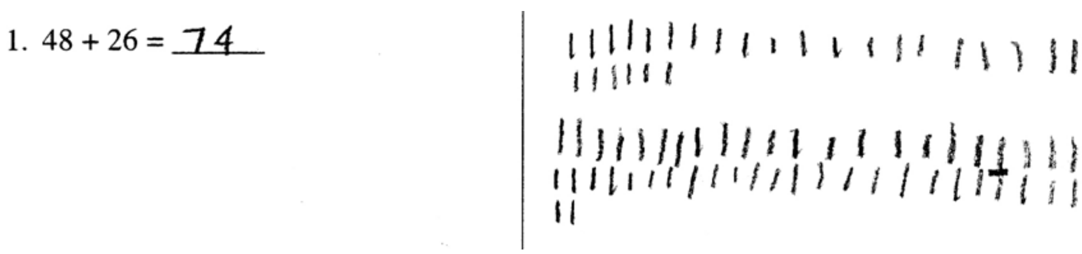
1. Reading the problem	Reading
2. Comprehending what is read	Comprehension
3. Carrying out a transformation from the words of the problem to the selection of an appropriate mathematical strategy	Transformation
4. Applying the process skills demanded by the selected strategy	Process skills
5. Encoding the answer in an acceptable written form	Encoding

The five questions the teacher asks clearly link to the five processes involved in solving a written mathematics problem. By using these questions consistently in class students develop a way of monitoring their progress towards answering written mathematics problems.

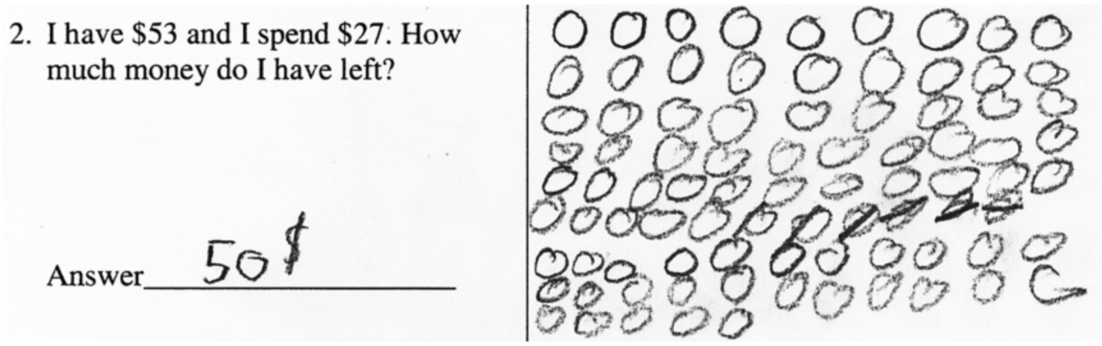
Research carried out in Australia and Southeast Asia suggests that about 50%–60% of students' errors in responding to written numeracy questions occur before students reach the process skills level (Clements, 1980; Marinas & Clements, 1990; Newman, 1977). In contrast, most remediation programs focus solely on the process skills.

What the assessment shows

To be considered to be on the place value framework, that is to be at the pre-place value level on the framework, a student must be able to count on and back. Some responses to the questions on the assessment can suggest that students are not using any sense of tens and ones. The following are examples indicative of a student who would not be considered to have reached level 0 on the place value framework.

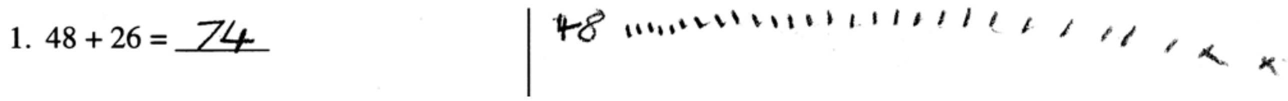


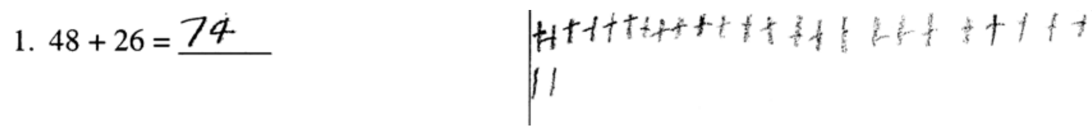
The response shown above from a Year 3 student is typical of a perceptual counter. That is, a student who has to recreate a number before operating with it. Both the 26 and 48 have been recreated using individual marks before counting from one to find the total. A single response does not indicate that this is all the student can do. However, a pattern of similar responses increases your confidence in determining the level at which a student is operating. The same student's response to Question 2 is shown below.



This response suggests that the student has attempted to draw 53 circles, lost track of the count and drawn an extra six circles before crossing these out. Creating 27 circles bears no relationship to determining the answer. Needing to reconstruct the numbers by ones suggests that this student has not yet achieved pre-place value and this student would be a member of the target group for the *Taking Off With Numeracy* program.

Counting on and back requires starting from one number and then either counting on the second number to find the total or counting back to find the difference between the two numbers.





The above responses of counting on by ones from 48 suggest that these students appear to treat 48 as standing in place of counting 48 items. However, no real use is made of the structure of tens and ones in determining the answer. When students have developed knowledge of tens and ones, ten is treated as a special unit. The use of ten as a special unit can manifest itself in one of two different ways. One method involves the tens and units being split off and handled separately. For example, in the following response to 48 + 26 the answer is achieved by splitting the tens and units.

1. 48 + 26 = 74

40+20=60
6+6=12+2=14

60 +
14
74

In combining the units (8 + 6) the student has also made use of double 6 by partitioning the 8 into 6 and 2. This is an example of the split method as is the response below.

1. 48 + 26 = 74

40 + 20 = 60
8 + 6 = 14

The following response shows the split method used with subtraction.

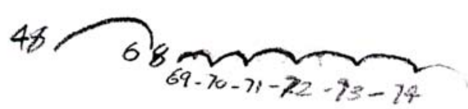
2. I have $53 and I spend $27. How much money do I have left?

50-20+3-7

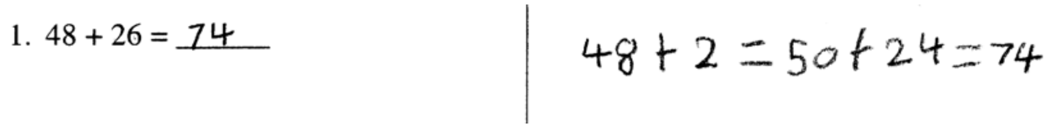
Answer $26

The second of the two methods arrives at the total by taking one number as the starting point and increasing by jumps of tens and ones (48 + 20 + 6). Sometimes the ones are broken into smaller hops as in the following response.

1. 48 + 26 = 74



There are several variations of the jump method. The following example starts with 48 and goes through 50 with a hop of 2 before jumping the remaining 24 to achieve the answer.



Within the project we take any solution method that effectively adds on or subtracts from one number that has not been split, as an example of the jump method. Consequently, subtracting 7 or jumping back 7 followed by subtracting 20 is also considered to be an example of using tens and ones with the jump method.

To use a multi-unit understanding of tens and one without relying on counting by ones, students usually need to develop part-whole knowledge of number combinations to at least twenty. When students learn to use trading with the traditional subtraction algorithm, subtraction problems such as 53 – 27 are transformed into problems involving subtraction within 20 (13 – 7 in this example).

$$\begin{array}{r} {}^{4}\cancel{5}\,{}^{1}3 \\ -\;2\;7 \\ \hline \end{array}$$

The subtraction algorithm commonly used in Australia has a standard way of regrouping the 53:

```
  53
 /  \
40   13
```

When this standard regrouping is used, students need part-whole knowledge of numbers to 20.

Teaching strategies

The teaching activities used in the program are designed to assist students in making the transition from a dominant use of count-by-one methods. The assessment identifies the most advanced modes of operating the students currently use and the teaching then focuses on developing non-count-by-one methods, and making their use explicit.

Teaching sequences address structuring within twenty (Ma, 1999), using the empty number line and other modes of recording (Gravemeijer, Cobb, Bowers, & Whitenack, 2000), as well as using a form of procedural variation (Gu, Huang, & Marton, 2004) to create effective scaffolds.

What makes this program effective?

Taking Off With Numeracy (TOWN) builds upon the work of the past ten years in the *Counting On* program (http://www.curriculumsupport.education.nsw.gov.au/primary/mathematics/numeracy/countingon/index.htm) and uses essentially the same learning framework employed in *Math Recovery* and *Count Me In Too*.

The program:

- builds on multiple research studies carried out in many different countries as well as cross-cultural analyses of teaching,
- uses a conceptual analysis of students' work rather than only a perceptual analysis,
- invests in helping the classroom teacher to more effectively address the needs of all students within the class. By investing in the professional knowledge of the teacher, the total resources of the school or system grow.
- is independently evaluated and continues to grow as it uses the results of the evaluation and design research to improve the effectiveness of the mode of delivery, and to shape the content of the program.

Although the program is successful in the Australian context, this does not mean that it will necessarily be as effective in a different culture. Successful implementation in a different context would require thoughtful planning and developing an effective local model of implementation.

How could a program like TOWN be implemented in other APEC economies?

It is important to recognise that the most effective implementation of a program has often been achieved through several cycles of 'customisation' to improve the fit of the program to its intended purpose. Before considering how to implement a program like *TOWN*, it is necessary to determine if the same problem exists. That is, is there a need for a program that addresses multi-unit place value and moving students on from count-by-ones strategies in your economy?

The current implementation model for TOWN makes use of video-conferencing and internet-based exchanges using small personal video cameras. However, this form of implementation leverages existing infrastructure within schools in Australia. Differences between economies in the design and delivery of curriculum as well as differences in the availability of resources suggest the need for different modes of implementation, as the most effective programs are designed to develop local capacity and transfer 'ownership' of the program.

All program materials are available in English and make use of both print and digital resources (see http://www.takingoffwithnumeracy.com.au). The main costs are associated with teacher professional learning and developing a local model of implementation.

References

Carpenter, T. P., & Moser, J. M. (1982). The development of addition and subtraction problem solving skills. In T. P. Carpenter & J. M. Moser (Eds.), *Addition and subtraction: A cognitive perspective.* (pp. 9-24). Hillsdale, NJ: Lawrence Erlbaum.

Chandler, C. C., & Kamii, C. (2009). Giving change when payment is made with a dime: The difficulty of tens and ones. *Journal for Research in Mathematics Education, 40*(2), 97-118.

Clements, M. A. (1980). Analysing children's errors on written mathematical tasks. *Educational Studies in Mathematics, 11*(1), 1-21.

Fuson, K. C. (1988). *Children's counting and concepts of number.* New York: Springer-Verlag.

Gravemeijer, K., Cobb, P., Bowers, J., & Whitenack, J. (2000). Symbolizing, modeling, and instructional design. In P. Cobb, E. Yackel & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms.* (pp. 225-273). Mahwah, NJ: Lawrence Erlbaum Associates.

Gray, E. M. (1991). An analysis of diverging approaches to simple arithmetic: Preference and its consequences. . *Educational Studies in Mathematics, 22*, 551-574.

Gu, L., Huang, R., & Marton, F. (2004). Teaching with variation: A Chinese way of promoting effective mathematics learning. In L. Fan, N. Y. Wong, J. Cai & S. Li (Eds.), *How Chinese learn mathematics: Perspectives from insiders.* (Vol. 1). Singapore: World Scientific.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States.* Mahwah, NJ: Lawrence Erlbaum Associates.

Marinas, B., & Clements, M. A. (1990). Understanding the problem: A prerequisite to problem solving in mathematics. *Journal of Science and Mathematics Education in South East Asia, 13*(1), 14-20.

Newman, M. A. (1977). An analysis of sixth-grade pupils' error on written mathematical tasks. *Victorian Institute for Educational Research Bulletin, 39*, 31-43.

Ross, S. H. (1989). Parts, wholes and place value: A developmental view. *Arithmetic Teacher, 36*(6), 47-51.

Steffe, L. P., Cobb, P., & von Glaserfeld, E. (1988). *Construction of arithmetical meanings and strategies.* New York: Springer-Verlag.

Wright, R. J. (1991). The role of counting in children's numerical development. *The Australian Journal of Early Childhood., 16*(2), 43-48.

Wright, R. J., Martland, J., Stafford, A. K., & Stanger, G. (2002). *Teaching number: Advancing children's skills and strategies.* London: Paul Chapman Publications/Sage.