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Collaborative Studies on Innovations for Teaching and Learning Mathematics in Different Cultures (II) - Lesson Study focusing on Mathematical Thinking

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PREFACE

We are pleased to present this progress report on the APEC project ***"Collaborative Studies on Innovations for Teaching and Learning Mathematics in Different Cultures (II) - Lesson Study focusing on Mathematical Thinking-".*** The papers were delivered during the APEC – KHON KAEN International Symposium and works.

At the third APEC Education Ministerial Meeting held on 29-30 April 2004 in Santiago, the ministers defined the four priority areas for future network activities. "Stimulating Learning in Mathematics and Science" is one of the four priority area. Based on this priority, the APEC HRD 03/2006 project "A Collaborative study on innovations for teaching and learning mathematics in different cultures among the APEC Member Economies" was approved by APEC Member Economies in August 2005. This project held meetings in January 2006 at Tokyo in Japan and June 2006 at Khon Kaen in Thailand. The project was managed by the Center for Research in Mathematics Education (CRME) in Khon Kaen University and the Center for Research on International Cooperation in Educational Development (CRICED) in University of Tsukuba.
Based on the success, the specialists from APEC economies decided to continue the project more four years in relation to following topics: Mathematical Thinking (year 2007), Communication (year 2008), Evaluation (year 2009), and Generalization (year 2010). The first three topics are selected in relation to three Lesson Study processes, Plan (for Mathematical Thinking), Do (for Communication) and See (for Evaluation). The result of each year will be based for the following year project. In the final year, Generalization will be set for the benefit of all subjects in education.

For year 2007, the APEC HRD 02/2007 project "Collaborative Studies on Innovations for Teaching and Learning Mathematics in Different Cultures (II) - Lesson Study focusing on Mathematical Thinking-" was approved in October 2006. For APEC member economies, Mathematical Thinking is a necessary ability for science, technology, economical life and development. Developing this ability in school is an important role of school in all economies.

With Lesson Study approaches, the project aims to

1) Collaboratively share the ideas and ways of mathematical thinking which is necessary for science, technology, economical life and development on the APEC member economies, and

2) Collaboratively develop the teaching approaches on mathematical thinking through Lesson Study among the APEC member economies.

As the goal of project, we would like to publish the report (or book) with CD-roms including the video of good teaching practices for developing mathematical thinking for teacher education in APEC economies and the world. In order to achieve the goals of the project, activities will be implemented in four phases

Phase I, A workshop and a Lesson Study meeting (a kind of workshop for specialists) among key mathematics educators from APEC member economies hosted by Center for Research on International Cooperation in Educational Development (CRICED), University of Tsukuba, Japan will be organized in order to share the idea and ways of mathematical thinking on curriculum level and teaching level (at Tokyo & Sapporo, December 2-8, 2006).

Phase II, Each co-sponsoring APEC member economy will engage in the Lesson Study project for developing some topics of mathematical thinking (February-July 2007).

Phase III, An International Symposium and a Lesson Study meeting (a kind of workshop for general teachers) will be organized in order to share teaching approaches for developing mathematical thinking by economies. The symposium will be hosted by Center for Research in Mathematics Education (CRME), Faculty of Education, Khon Kaen University, Thailand (at Khon Kaen, August 16-20, 2007).

Three hundred seventy-nine participants and observers attended the symposium. Three hundred forty-nine local participants and observers, including university lecturers, mathematics teachers, experts and educational policy makers related to mathematics education in Thailand, and thirty participants and observers from fourteen member economies of APEC, including Australia, Brunei Darussalam, Chile, Chinese Taipei, Hong Kong, Indonesia, Japan, Korea, Malaysia, the Philippines, Peru, Singapore, USA, and Vietnam and participants from other countries such as UK, Lao PDR and South Africa attended this meeting.

Phase IV: The 'APEC Workshop on: Improving the quality of the mathematics lesson through Lesson Study' was held in Thailand in 15-16 August 2007. In this workshop, Japanese teaching method was proposed to teachers of Thailand in the style of workshop on Lesson Study. Teachers from the Attached Elementary School of the University of Tsukuba, Japan came to Thailand to demonstrate two phases of Lesson Study – teaching Thai students in the real classroom and reflecting on teaching with Thai teachers. Activities in this phase reflected the title of the project. In addition, these activities were also effective in supporting the movement, which was developed in northeast area of Thailand by Khon Kaen University.

To disseminate the knowledge of lesson study shared by the APEC member economies at the APEC – KHON KAEN International Symposium, we are publishing this book of their reports and VTRs of Lesson Studies.

We are indebted to the Office of Commission of Higher Education, Ministry of Education and Khon Kaen University for their full support for the APEC project *"Collaborative Studies on Innovations for Teaching and Learning Mathematics in Different Cultures (II) - Lesson Study focusing on Mathematical Thinking-"*. More importantly, we would like to thank all members of CRME and staff of the Faculty of Education for their contributions in organizing the symposium and completing this progress report. Finally, we would like to use this space to express our gratitude to our keynote speakers: Alan J. Bishop, Kaye Stacey, David Tall, Fou-Lai Lin, Shizumi Shimizu, Masami Isoda, Abraham Arcavi, Akihiko Takahashi and all specialist for continuing to contribute to this APEC project.

October, 2007
APEC Project Overseers
Maitree Inprasitha and Suladda Loipha (Khon Kaen University, Thailand)
Masami Isoda and Shizumi Shimizu (University of Tsukuba, Japan)

Contents

HOW TO DEVELOP MATHEMATICAL THINKING

Shizumi Shimizu, University of Tsukuba

1. Thinking mathematically and mathematical thinking

(1) Idea of Mr. Kenzo Nakajima

Who introduced mathematical thinking into the Course of Study revised in 1958 as aims of mathematics in Japan. Creative activities to be good for mathematics nearly equal to 'thinking mathematically' Mathematical thinking in the Course of Study revised in 1958; aims of elementary school mathematics.

In the aims, mathematical thinking located in the two phases

Mathematical thinking as results created by students

Mathematical thinking as tools students use adequately

There was developing of a scientific attitude in the background.

Mathematical thinking as one point of view of evaluation (after 1970's)

→ development of mathematical thinking

→ meaning of 'development' became ambiguous

→ need to realize the two phases of mathematical thinking again

(2) Idea of Mr. Shigeo Katagiri

His life work is to analyze and classify mathematical thinking from 1960's

Mathematical thinking consists of mathematical idea, method, and attitude which support thinking mathematically

2. Developing student's mathematical thinking in classroom

(1) Putting student's activities in the center of classroom and these activities to be creative or inventive for students

A lesson (classroom) develops mathematical thinking by students' problem solving. Teachers guide and support their activities.

(2) Creative activities (problem solving) should be meaningful both for students and teachers.

We try to analyze the elements and structures of mathematical thinking and to help students acquire them.

3. Thinking mathematically

(1) Students' independent activities

Engaging oneself, not other people's activities

(2) Motivations and phases of activities

Engaging mathematical activities according to phases of them adequately

Motivations and phases

・ from Need in life, Explanation of phenomena

→ Using mathematics

Considering or judging by using mathematics

・ from Intellectual curiosity, Pursuit of mathematical beauty

→ Creating or discovering mathematics

Thinking creatively or extensively and discovering or inventing new facts, skills, ideas etc. By relation with mathematics of experience by everyday life and having learned already.

・ Supporting using and creating mathematics, and from pursuit conclusive evidence or enrichment

→ Explanation or verification

Necessity for understanding by oneself, persuading other peoples sharing results each other, and refining them better

4. Observing classroom activities

5th grade sum of interior angles of polygons

5. Some points of view for improving math classes

(1) To help students make thinking mathematically a custom

(2) To represent students' inner process of thinking mathematically

(3) Grasp results exactly from thinking mathematically or mathematical problem solving

(4) Two adequacy for posing problem

・ to be good for aims of lesson

・ to lead results to be good for problem posed

(5) Problem posing and the result of solving the problem posed

・ to consider the characteristic of problem; self-creating aspect

(6) Developing the mind of challenge, confidence, feeling of effectiveness

(7) Collaboration and creating

→ Japanese proverb; sann-ninn yore-ba monnjyu –no chie

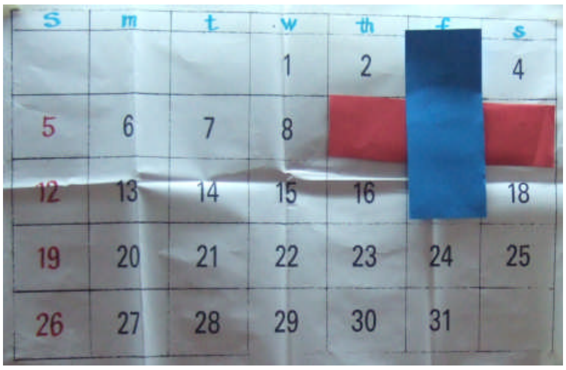
How to develop Student's Mathematical Thinking in Classroom

Objective: 1) To find and think about average 2) To develop mathematical thinking and children's image from surroundings.

Field note：

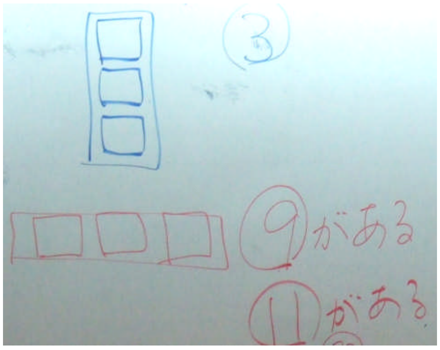
13:24～
T This is Japan's map. Around country is sea, I will give this to your teacher. Also these color papers. I will give you a card with Japan's view.

13:30
T. I will stick this paper. Watch it carefully.
What it is?
C. Calendar
S. How to say Sunday?
C. Sunday…Saturday
T. I will hide somewhere.
C. It looks like Hospital sign.

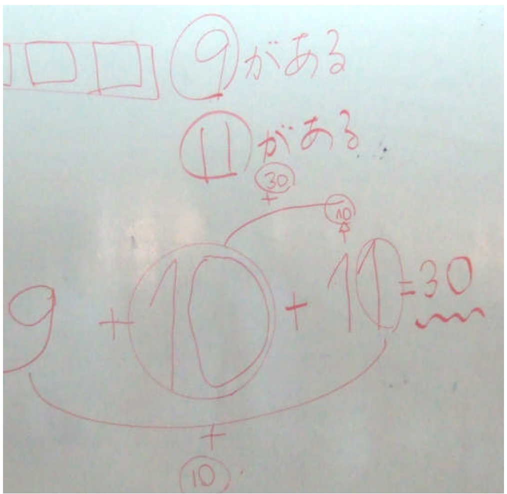


T. Summary of Blue is bigger or Red is bigger?
Blue 3+7+14 Red 6+7+8
Almost C. answer Red is bigger than Blue

T. Why do you think so?
C. Cause it has 9 so it's bigger
T. It has 3 numbers
T. I will ask you again. Summary of Blue is bigger or Red is bigger?
Nobody answered Blue is bigger
C. Because Blue has 3, but red has 11
T. We will check it by calculate them

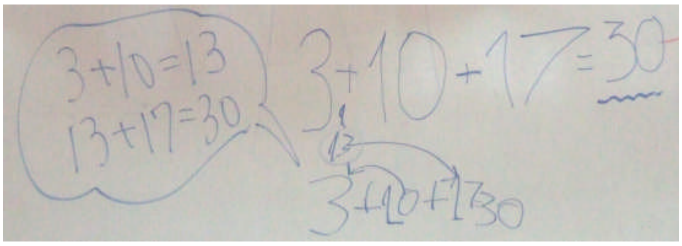


T. Had already learnt addition right?
T. Blue is 3+10+17 right?
T. Red is 3+10+17 right?
C. Wrong!!
T. Can you correct it?
C. 9+10+11
T. What is the answer?

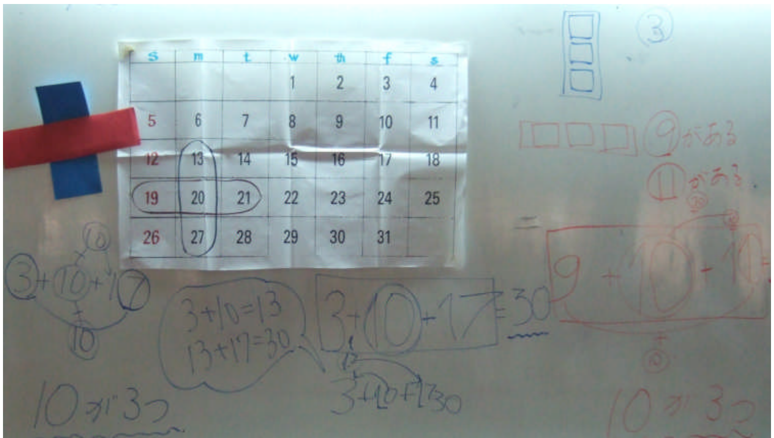


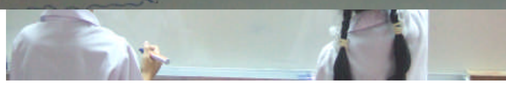
C. Blue 30.Clap. Red 30.clap
T. Blue is 30. Red is 30. They are same.
T. How do you calculate the red one?
C. 9+11 is 20 then 20 + 10 is 30
T. Can you draw the lines?
T. Look it carefully. Do you have other way to calculate?
C. 3+10+17. 10+3 is 13 then 13+17 is 30

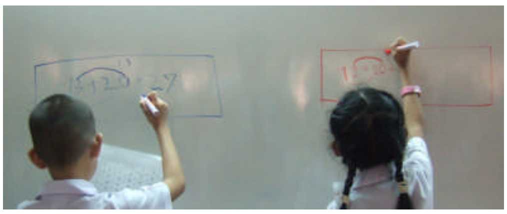
13:52
T. She/he calculates from left to right. But another he/she did the faster way to calculate. It is easier to calculate, when we have three of tens. Can we do like this in that one.
T. discuss with your friend
C. Draw the line 3 and 7 of 17
T. 10, 10, and 10 are 30
Both have there of tens



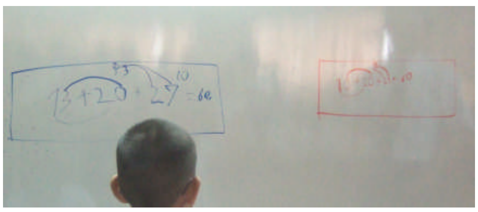
We can calculate faster if make there of tens.
10 is central, inside the card.
If we can find this middle number, we can calculate faster.
T. I'll change cards position.
What are summary of them?
Blue: 13+20+27 a lot
Red: 19+20+21 a little
Same: a little
T. Please. Calculate them.
C. Same
T. What it is?
C. 60
T. Writes the equation please.



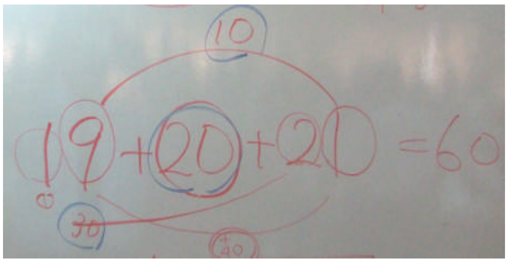




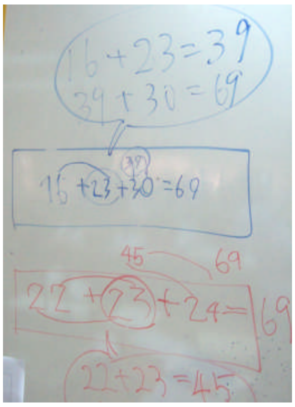
14.02
T. It is really to be 60?



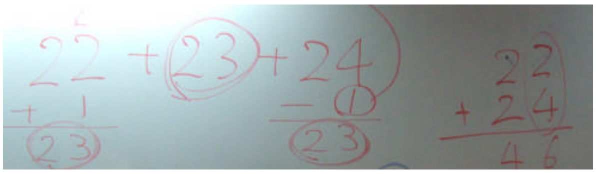
14.07



T. Is it possible to leave the central number?
C. Yes. 19+20+21, 19+1 is 20. 20+20+20
T. What about blue one.
C.13+20+27, 3+7 is 10. 20+10 is 30
T. Good Job! It faster to calculate when leaves central number.



14.12
T. If on the calendar, wherever we put cards it will be same?
T. Anywhere you want to put cards?
C. 16+23+30
C　赤：
T　同じになる：　違う：　分からない：
T. I will take cards off and circle around, instead.



WHERE DO MATHEMATICS PROBLEMS COME FROM IN ELEMENTARY SCHOOL CLASSROOMS?

The Problem Solving Approach in the classroom: a dialectic with the conflicts and comprehension among students

Masami Isoda, CRICED, Univertisy of Tsukuba, Japan

The book (Isoda, 1996) was written to support elementary school teachers in Japan who plan lessons based on the Problem Solving Approach, which is a renowned approach for teaching mathematics around the world. According to mathematics education theory regarding lesson plan development or textbook sequencing, mathematics educators usually take account of the sequence of mathematical content, a range of situations including real-life examples, and mathematical representation for the process of abstraction. For example, one of those embedded in the textbook based on the ‘Model of, Model for’ framework by the Freudenthal Institute is ‘Mathematics in Context’, which includes the process of Situation, Model and Form through Mathematization.
Across the world there are different textbooks, based on the local curriculum. However, most of these textbooks do not directly deal with students’ misunderstandings. On the other hand, Japanese elementary school textbooks and teachers’ guide include expected children’s answers for each problem and suggestions for how to treat children’s misunderstandings in class based on the experiences of lesson study. This is possible because the Japanese curriculum is national and textbooks are shared. When curriculums vary across different schools and classrooms, it is not easy to share these kinds of ideas.
Many elementary school teachers and mathematic educators believe that mathematics problems arise from daily situations. Isoda (1996) discussed an alternative idea based on Japanese tradition that mathematics problems arise from problematic situations for children in special occasions during lessons that follow the curriculum sequence. In the Japanese Problem Solving Approach, known since the 1 960s, a problem situation is defined as confronting the unknown when compared to what has previously been learned. Problems posed within the teaching sequence, developed via curriculum planning, enable children to learn mathematics based on prior learning. What is theoretically new in this book (developed early 1990s), are the following. The book describes the source of problem situations through the process of extensions originated from the curriculum sequence; it explains the development of conceptual and procedural understanding through the learning of mathematics based on the curriculum sequence; and it likewise explains how to utilize dialectic discussion among classroom (Neriage), which involves each other’s perspective.

Library data for your reference:
Masami ISODA edited (1996), Problem-Solving Approach with Diverse Ideas and Dialectic Discussions: Conflict and appreciation based on the conceptual and procedural knowledge, Tokyo:Meijitosyo Pub. (written in Japanese) (English Ver.6 for the Preparing Publication.

Chapter 1
The Lesson Structure Based on a Problem-Solving Approach that Produces Diverse Ideas and Promotes Developmental Discussions: Focusing on the Gap between Meaning and Procedure

The lesson planning based on the theory of understanding on curriculum sequence

Masami Isoda, University of Tsukuba

In an introductory lesson on adding fractions with different denominators that aims to teach children how and why they should perform calculations like 1/2 + 1/3, children who do not know the meaning of 1/2 L (ℓ) or 1/3 L cannot objectively understand the meaning of the word problem. Children who are not proficient in the procedures of reducing fractions to a common denominator, previously learned, will likely struggle with solving problems. Teachers will be well aware of the importance of meanings and procedures (including form and way of drawing) learned over the course of problem-solving lessons.

The Japanese Problem Solving Approach usually begins from children's challenges of a big problem based on what they have already learned. This chapter will use specific examples to show that previously learned meanings and procedures (form and way of drawing) help elicit a variety of ideas (conception) from children. Then it will describe methods of creating lessons that support children's learning through the eliciting of diverse ideas (even if it is misunderstanding) and a developmental discussion (a dialectic among students). This is based on the notion that it is precisely when people are perplexed by something problematic that they develop their own questions or tasks, have a real opportunity to think about these, can promote their own learning, and can reach a point of understanding. The following aims to shed new light on the true significance of this notion.

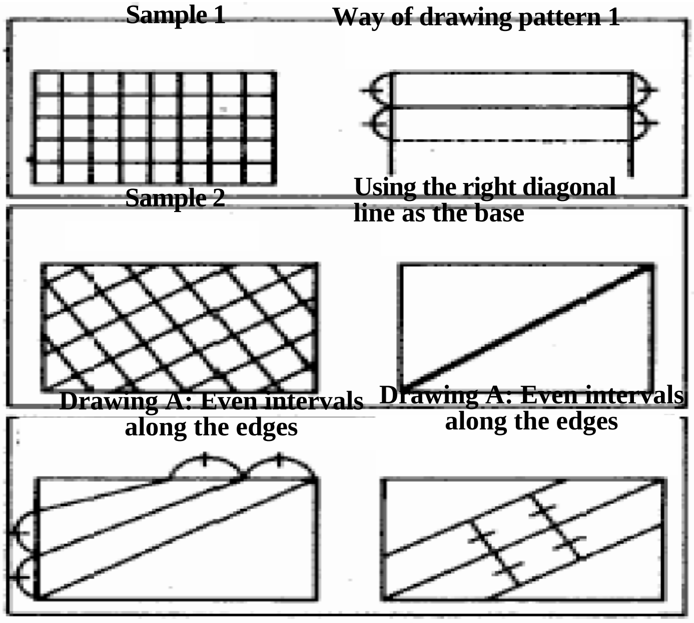
1. It goes well! It goes well!! What?

In Japan many teachers have experienced the following situation. The teacher finishes a class feeling confident that the lesson went well and believing that the children understood the material, but the children say "What? I don't understand" in the very next class. The student comments clearly indicate that they had not developed a good understanding of the material previously presented, even if they said they had clearly understood it at that time. This is precisely the treasure secret of the problem-solving approach: to elicit diverse ideas including misunderstanding and promote developmental discussions.

First, let us examine this approach by taking a look at a fourth grade class taught by Mr. Kosho Masaki, a teacher at the Elementary School attached to the University of Tsukuba (Sansuka: mondai kaiketsu de sodatsu chikara, Toshobunka 1985).

1-1. Fourth Grade Class on Parallelism Taught by Kosho Masaki

To introduce parallelism, Masaki started by drawing a sample lattice pattern. The following process shows how children developed the idea of parallelism in his lesson study.



<u>Task 1. Let's draw the Sample 1 lattice pattern</u>

All of the children were able to draw this lattice pattern by taking points spaced evenly apart along the edges of the drawing paper and drawing lines between them. "It went well!"

<u>Task 2. Let's draw the Sample 2 lattice pattern</u>

The children began to draw the pattern based on a diagonal line moving upward to the right. What kind of reactions do the children have? The results are varied and depict several different strategies. However, they can generally be categorized into the ways shown in Drawings A and B.

<u>Developmental Discussion: "What?", What happened in Task 2?</u>

Masaki explained his problem solving approach as follows. In this (dialectic) situation, the children, even those who completed the task mechanically, were asked why they were able to draw the pattern in Task 1 but not the pattern in Task2. They were asked to try to find various ways in which to draw lines in order to reproduce the pattern shown in Samples 1 and 2. Because the children saw that others came up with results different from their own and everyone grew in confidence from their ability to draw the pattern, they began asking one another "How did you draw that?" and "Why did you think you could draw it by doing it that way?" They found it necessary to discuss their results. They began to distinguish between methods and to develop explanations. Through this developmental discussion, they were able to produce the word 'parallel' for what they had found based on what they had learned from others.

When children become aware of the unknown – in other words, there is a gap in their knowledge or meet different ideas – they become confused and think "something is wrong." This is then followed by a sort of conflict, leading to the questions "What?" and "Why?" Furthermore, when children enter developmental discussion (a dialectic) and are faced with ways of thinking that are unknown to them (knowledge gaps with others), it also causes conflict, forcing them again to ask "What?" and "Why?" Here again, they have to compare their way of thinking with that of others, evaluate it again by themselves and discuss their findings with other children. In this sequential flow, children make use of what they previously learned to turn the unknown into newly learned knowledge (a new understanding). This is the problem-solving approach discussed in this book based on conflict and understanding.

Here, one must ask why then did all the children feel that the drawing in Task 1 had "gone well," but in Task 2 two distinctly different types of drawing appeared. The reason lies in the diverse ways of thinking that appear in the sequence of tasks. In the next section of this chapter, we will clarify this using the terms 'conceptual or declarative knowledge' and 'procedure (form and way of drawing).' Then based on these terms, the sequence of tasks is analyzed again.

1-2. Looking at Masaki's Class in Terms of Meaning and Procedure

Meaning (in this instance, conceptual or declarative knowledge) refers to contents (definitions, properties, places, situations, contexts, reason or foundation) that can be (re)described as " ~ is ..." For example, 2 + 3 is the manipulation of '004—000'. The meaning can also be described as: "2 + 3 is 0 04—0 00" and as such explains conceptual or declarative knowledge. In Mr. Masaki's class, this method can be used to explain as follows: "The sample model is parallel lines." It therefore describes the meaning, which subsequently becomes the foundation of creating conceptual or declarative knowledge regarding the parallelism of the sample model.

Procedure (in this instance, procedural knowledge) on the other hand refers to the contents described as "if...., then do..." This is the procedure used for calculations such as mental arithmetic in which calculations are done sub-consciously. For example, "if it is 2×3, then write 6" or "if it is 2 + 3, then write the answer by calculating the problem as 0 04—0 00." This is procedural knowledge.

By doing this, you may say, "Oh, I see, the meaning is merely another expression of the procedure, that's why they match." Yes, that is true for those who understand that they do match. However, people do not immediately understand that they match. Even if they know that the sample models are graphs of parallel lines (conceptual knowledge), this does not mean that they can draw them (procedural knowledge). On the other hand, even if people can draw parallel lines (procedural knowledge), it does not mean that they understand the conceptual meaning (properties, etc) of parallelism. Cases when conceptual and procedural knowledge do not match are not only evident in mathematics classroom, but also in other facets of everyday life. For example, despite knowing their alcohol limit (conceptual knowledge), there are cases when people drink too much. Furthermore, it is this mismatch and contradiction that becomes the catalyst for the process in which people encounter a conflict, experience reflection, deepen their knowledge and gain understanding.

Let us return to Masaki's class. At first glance, the way of drawing pattern 1 in the first task appears to be a general method for drawing figures. However, from the and B in task 2, it seems that the children confused the two procedures shown in the box. Even if the children produce the same problem, how they acquire conceptual and procedural (form and way of drawing) knowledge, and the use of that understanding and knowledge are many

<u>Way of drawing 1: Procedure a</u>
→Way of Drawing A; Task 2
If you want to draw the model, draw lines spread perspective of the ways shown in Drawing A evenly apart from the top edge of the paper.
<u>Way of drawing 1: Procedure b</u>
→Way of Drawing B; Task 2
If you want to draw the model, draw lines spread answer, the ways they understood the evenly apart.

and varied.

Based on analysis of the ways shown in drawings A and B, Masaki's class is described by conceptual and procedural knowledge.

The gap between the Sample model (conceptual knowledge) and the way of drawing (procedural knowledge): encounter a conflict

- Thinking "hold on, I can't draw this using procedure a; the lines cross over if extended, but as shown in the samples, the lines do not cross."
- "Why was I able to draw Sample 2 pattern using procedure b and not procedure a?"

Reviewing the way of drawing (procedure), and revising and reconsidering the semantic interpretation of the Sample model, which acts as the foundation of the drawing method.

- "How did you draw that? Why did you think it would work out if you did it that way?"
- Reason (coming from semantic interpretation of the Samples); lines in the Samples are
  all evenly spread apart, so they don't cross over.
- "I tried to draw the lines spread evenly apart, but they crossed over. How should I do it?"
- How do you properly draw lines spread evenly apart? By using the correct drawing
  method, which makes right angles and alternate interior angles evident.

Elimination (bridging) of the gap between the semantic meaning and way of drawing (procedure): to a coherent understanding

Taking into meaning (even spreading of lines, no crossing-over, and characteristics of right angles, corresponding angles and alternate interior angles), designation (definition) of parallel and drawing method (procedures including equal spread of lines, right angles, corresponding angles and alternate interior angles).

Within the developmental discussion process, procedure b, in which lines are drawn equidistantly at all points, works for both Samples 1 and 2. In contrast, procedure a, in which the lines are drawn from the top edge of the paper, clearly works for Sample 1, but does not work for Sample 2. Because Sample 1 is contrasted with Sample 2, the meaning of equal spread of lines is connected to the method of drawing with attention to lines equidistant at all points, right angles, corresponding angles and alternate interior angles. As a result, the basis (meaning) of why that way of drawing was attempted is explained by the children's comments.

Naturally, Masaki anticipated and expected to encounter undifferentiated schematic interpretations and drawing methods on the part of the children, and as such planned his classes accordingly. The teacher does not start by teaching the meaning and way of drawing parallel lines he is familiar with, but in fact starts by teaching at a level which assumes that children have not yet learned the word 'parallel.' The teacher tries to make use of previously learned methods of drawing parallel lines (procedures) that the children already know. By confirming previously learned knowledge, the teacher instills a sense of efficacy through leading children to a successful completion of the task. Following that, the teacher then makes the children face the difficulties by questioning "What?" at times when it does not work well. Due to the conflict that arises, children then ask about the meaning of the parallel lines. The teacher aims to have the children create their own reconstruction of the method of drawing and the meaning, using what they already know as a foundation.

Looking back, it can be seen that the flowchart presented on the right is embedded in Masaki's class. As is indicated, the class is structured in such a way that the children proceed from a feeling that everything is "going well" to suddenly asking "What?". This transition serves as the context in which a diverse range of ideas appears regarding how the children have understood the problem and what type of meanings and procedures they have acquired. This class is indeed a type which solves problems through developmental discussion (a dialectic) and makes use of a diverse range of ideas by overcoming the conflict of "What?" sorting through and clearing up previously misaligned meanings and procedures, and finally reaching a stage of understanding.

Dialectic Structure of Mr. Kosho Masaki's Parallel

Confirming Previously Learned Knowledge
Situation: Task 1 "It goes well"—S**ense of Efficacy**
Even if gaps in meaning and procedures exist, they do not appear here.
Different Situation from Previously Learned Knowledge: Task 2
There are children who show gaps in their understanding of meaning and procedure and some who don't.
"What?" – **Conflict**
Developmental discussion (a dialectic) by questioning new meanings and procedures
Acquisition of a **Sense of Achievement** by Overcoming the Conflict and Proceeding **through Understanding**

2. Reading the children's diverse range of ideas through meaning and procedure (form and way of drawing)

For the planning of a lesson on the Problem Solving Approach, it is necessary to anticipate the diversity of children's responses and plan a developmental discussion for studying the target of the lesson. This section shows ways of reading and anticipating children's ideas using the words 'meaning' and 'procedure (form and way of drawing).' The theory of conceptual and procedural knowledge in mathematics education by James Hilbert (1986) is well known, and in Japan, Katsuhiko Shimizu applied a similar idea in classroom research (1986). Meaning and procedure for lesson planning theory has been developed by Isoda (1991) as an adaptation of cognitive theories to the progressive development of mathematics ideas within lessons.

To begin with, we would like the readers to read once more the above-mentioned explanation of meaning and procedure, and do the following exercise.

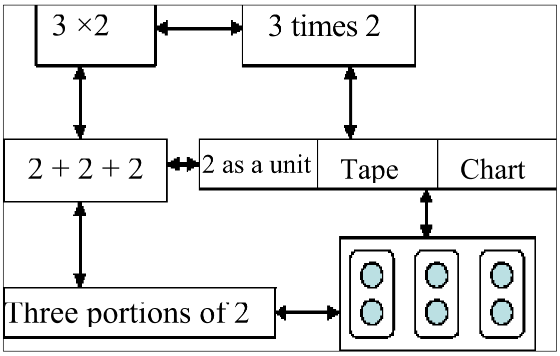
Exercise 1

Which do the following correspond to: meaning or procedure?

1. Reduction to the common denominator refers to finding the common denominator without changing the size of the fractional number.
2. <u>In order to</u> compare the size of fractional numbers, either reduce or increase the fractional number size.
3. <u>In order to</u> divide by a fractional number, take the reciprocal of the divisor and multiply.

2-1. What is meaning? What is procedure (form and way of drawing)?

a. What is meaning?



Meaning (conceptual knowledge) can be illustrated by "man is a wolf," for example. Of course, a man is a human being, but by likening man to a wolf and changing the way of saying it, one can make a sentence that aims to express the meaning of "man." The previous example "2+3 =004-000" gives a concrete example and changes the way it is said to express the meaning. The mathematical expression "3X2 = 2 + 2 + 2" also expresses meaning (in Japanese, 2X3 means 2 + 2 + 2). It is a rephrasing too. Such a rephrasing not only refers to a concrete example but also refers to what is already known. Note that the meaning of multiplication that children learn in the second grade can be summed up as shown in the figure above. The characteristics of the meaning are seen in the fact that a number of elements are connected like a net, and as such, we as teachers think that children can understand the meaning in more diverse ways when we are able to interpret like this. The important thing regarding diverse expression is that the meaning is in fact picked out and expressed through such rephrasing.

In response to the problem "How many L and dL is 1.5 L?", a student replied: "Before, we learned that 1 L is 10 dL, and that 1 dL is 0.1 L. If I use that, 1.5 L is 15 parts 0.1 L. 10 parts 0.1 L is 1 L. The remaining 5 parts are 5 dL. So, 1.5 L is 1 L and 5 dL." When that child explained the basis of her reasoning, we as teachers can see that the child has made a deduction and explained it based on meaning.

b. What is procedure?

Procedure (procedural knowledge, form, way of drawing, method, pattern, algorithm, calculation, etc.) can be expressed as follows: "If the problem is to divide by a fractional number (recognizing conditional situations), then take the reciprocal of the divisor and multiply." The first characteristic of procedure is being able to process automatically, without question, and instantly. However, proficiency (in other words, practice) is necessary. When answering the question how many dL are in 1.5 L, take a case where a student rapidly answers "1 L 5 dL." If the student automatically follows the rule "if L is interpreted as L and dL, then focus on the position of the decimal point and think of L as coming before it and dL as coming after it," then one could acknowledge that this student is using procedure. Being able to solve a problem instantly like this by using procedure means that we have come to a stage where we can find a solution without having to spend a lot of time deducing meaning, which in turn brings us to the point where we can consider reducing thinking time (e.g. short-term and working memory). Another characteristic of procedure is that it produces new procedures such as the complex grouping of the four operations, as seen in the example of division using vertical notation (long division) whereby numbers are composed (estimating quotient), multiplied, subtracted and brought down (to next lower digit). If each procedure is not acquired, it is difficult to use complex procedures that incorporate some or all of them. In other words, if one becomes proficient, it does not matter how complex the grouping of procedures are, as one will be able to instantly use them. Simplifying complex deductions and being able to reason about a complex task quickly means that one is able to think about what else should be considered.

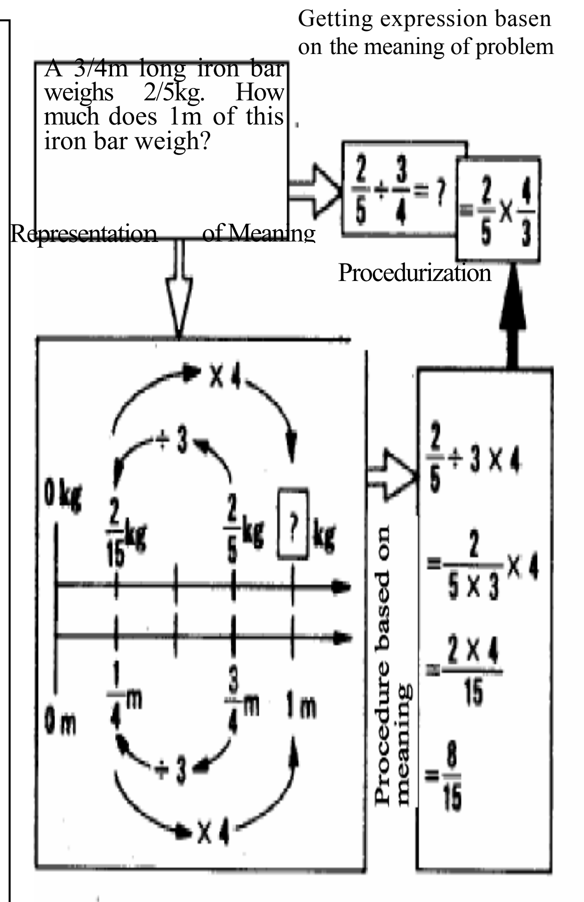
c. The relationship between meaning and procedure

As was shown in the method of drawing and the meanings of the patterns in Masaki's class, there are instances when the meaning and procedure match (no appearance of gaps, consistency of use) and other instances when they do not match (appearance of gaps, inconsistency). In learning process through the curriculum or planned sequence, there are situations where the meaning and the procedure contradict each other and situations where they do not. Moreover, from the curriculum/teaching-learning sequence perspective, these two instances are mutually linked or translated as follows.

<u>Procedures can be created based on meaning</u> (**the procedurization of meaning**, in other words, procedurization from concept). For example, when tackling the problem "How many L and
dL is 1.5 L?" for the first time, a long process of interpreting the meaning is applied and the solution "1.5 L is 1 L 5 dL" is found. Additionally, this can be applied to other problems such as "How many L and dL is 3.2 L?" with the answer being "3.2 L is 3 L 2 dL." Children soon discover easier procedures by themselves. Simultaneously, children realize and appreciate the value of acquiring procedures that reduce long sequential reasoning to one routine, which does not require reasoning.

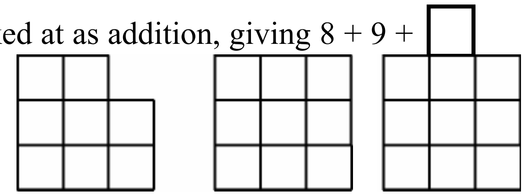
There is a remarkable way to shorten the procedure from known concept and procedure. The example, "if the problem is the division of fractional numbers, then take the reciprocal of the divisor and multiply" is shown in the diagram below. Using the previously learned concept of proportional number lines, the meaning of the calculation is represented and the answer is produced based on this representation. As a result of this representation, the alternative way of calculation 'take the reciprocal of the divisor and multiply' is reinterpreted so that it can be produced simply and quickly from an expression of division. Thus children reconstruct a procedure that can be carried out simply and quickly by reconsidering the result based on meaning. Even in a simple case such as the multiplication 2 times 3, this is $3 + 3 = 6$ as a meaning, but as a procedure, 3×2 is interchangeable with the memorized result of 6. This remarkable way is also the procedurization of meaning. Many teachers believe that the procedure should be explained based on meaning, but the alternative is often preferred because it is much simple and easier. Using one of the key values of mathematics, namely simplicity, we finally develop procedure based on meaning.

Meaning becomes the foundation for acquiring procedure. When children struggle to use previously learned knowledge, and if they employ a d versity of meanings for producing procedure, the importance of faster and easier procedures for obtaining answers will become clearer, as the alternative is to follow the difficult path of long reasoning. By debating diverse meanings in order to reason, children can clarify meaning and thus may recognize the situations for which the produced procedure is applicable. Procedure has the 'if, then' structure. The 'if' describes the conditions of applicable situations; when applicable, it is acceptable to carry out the 'then'. Negotiating meaning is important for understanding applicable situations, even if it is very difficult to clarify the conditions for applicability without extension (the notion of extension is explained later).



The above is an example of how procedures can be created based on meanings. However, the reverse can also be achieved: meaning can be created based on procedure (meaning entailed by procedure, in other words, conceptualization of procedure). Let us consider this notion from the perspective of addition taught in the first grade and multiplication taught in the second grade of school. In the first grade, as in the operation activity where '○○ ←○○' means 3 + 2, children learn the meaning of addition from concise operations and then become proficient at mental arithmetic procedures (the procedurization of meaning). At that point, calculations such as 4 + 2 + 3 and 2 + 2 + 2 are done more quickly than counting, which is seen as a procedure. Further, in the second grade, compared with the case of several additions, only repeated addition problems lead to the meaning of multiplication. It is here that the specific procedure known as 'repeated addition' (now called multiplication) is incorporated into the meaning (meaning entailed by procedure). The reason such situations are possible is that children become both proficient at calculations using addition and familiar enough with the procedure to do it instantly. Children also see the meaning of
a situation such as in the following picture showing three groups of objects. To find

the total number of objects, it can be looked at as addition, giving 8 + 9 + 10, but by moving an object from the third group to the first, it can be looked at as repeated addition or multiplication, giving 3×9. Children unfamiliar with the procedure resort to learning addition and multiplication at the same time, which in turn make it more difficult for them to recognize that multiplication can be regarded as a special case of addition.



Only people who have a good understanding of the meaning and the procedure use them as if they were one; they can be thought of as two sides of a coin, each of which has different features but together form the one coin[1]. On the other hands, from the curriculum sequence and its teaching-learning perspective, meaning and procedure develops mutually. Due to the fact that meaning can become procedure and vice versa in the teaching-learning process on the curriculum, only teacher can recognize the situation which meanings and procedures do not related mutually and plan how to develop mutual relationship. As this book aims to support teachers in their lesson planning, it is up to each teacher to decide what is meaning and what is procedure in each class in accord with the actual situation of the children and the classroom objectives.

2-2. Using meaning and procedure (form and way of drawing) to anticipate children's ideas

In the problem solving approach, teachers anticipate children's ideas in order to plan to develop their ideas using what is already known. Meaning and procedure support this anticipation[2].

a. Knowing meaning and procedure allows you to anticipate children's incomplete ideas

Some months after learning how to divide fractional numbers, children are asked: "Why does that happen?" Many children reply "because you turn it upside down and multiply" (procedure), even though they could answer with meaning when they first learned well about it. This indicates that they lose meaning in exchange for procedural proficiency (proceduralization of meaning). Here we would like readers to answer Exercise 2, keeping in mind children who tend to forget the meaning.

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[1] The metaphor is as same as Sfard but the idea itself developed originally at the publication in 1991 as a result of lesson study with elementary school teachers. She had pointed the same idea.

[2] In Japan, curriculum standards are fixed and textbooks are distributed by the government. One of the basic curriculum sequence and textbook contents sequence in Japan is 'extension' or 'expansion', that is, extending learned procedures to new situations. Depending on the situation, teachers can share children's responses through the Lesson Study and teachers' guidebooks, and at the same time, they can anticipate children's reasoning and the process of discussion.

Exercise 2 Expressing a number with one denomination in a form that uses multiple denominations

A third grader with previously learned knowledge able to quickly give the answer "1.5 L = 1 L + 5 dL" is asked the following question: "4.2 m = how many m and how many cm?" Anticipate the child's reaction.

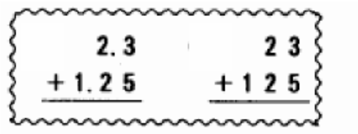
A procedure that a child becomes proficient in is typically swimming or riding a bicycle; it is not easily forgotten, but meaning does not stay in one's consciousness unless it needs to be used. The most common answer by children to the above exercise, as expected, is "4.2 m = 4 m + 2 cm." In the third grade, children are taught to work as far as the first decimal point in small numbers. Therefore, when learning, children are usually only faced with units of 1/10 such as in L and dL, or cm and mm. Children who become able to quickly give the answer "1.5 L = 1 L + 5 dL" only experience the situation where that procedure is applicable. As a result, they become unable to make semantic judgments on when that procedure can be used.

The correct procedure "If ..., then..." will always produce the correct result as long as the conditional "if" part of the semantic judgment is correct. However, when children only experience applicable instances, they over-generalize the meaning and become unable to make a correct judgment. As a consequence, many children who use this so-called 'quick/instant' procedure may use it in cases where it does not apply.

It should be noted that this quick response procedure is not only something that the teacher has taught, but rather is an extremely convenient idea that the children may have arrived at on their own. Even if this concept is invalid, children will not recognize this as long as they continue to be presented with tasks that do not show the weaknesses of the invalid concept. For example, even if children from Mr. Masaki's class completed the first task using an invalid concept, the underdeveloped nature of the concept would not become apparent until it was applied to another task. Therefore, what the teacher should first recognize is a child's idea created as his/her own. From there, the next step is to deepen that idea by investigating whether or not that idea can be generalized to other tasks. This is the challenge for teachers.

b. Gaps between meaning and procedures appear in extending situations

As presented at the beginning of this chapter, the steps “It goes well! It goes well! What?” are important. As long as everything goes well and is applicable in the end, the gaps between meaning and procedure will not become a problem. In such a situation, children are not faced with a difficult situation; they are within the range of previously learned knowledge, and have not yet been challenged by the unknown. However, a situation when something does not go well or when there is a need to close a knowledge gap is indeed one where true discoveries and creations are made. When a person thinks “What?” in a situation, this indicates issues that should be given genuine thought. An example of when things do not go well is the ‘extending situation.’’ In an extending situation[3], the gap between meaning and procedure appears as diverse ideas. Here, let us look at the example of the extension of a procedure from whole numbers to decimal numbers.



Example 1) shown on the right is an over-generalized idea that can be seen in the decimal number calculation. It is usually explained as misunderstanding the meaning of place-value.

Why does this type of idea appear? It arises because, when calculating with whole numbers using vertical notation as in example 2), the proper procedure is to write the numbers so that they are aligned on the right side. Example 1) indicates that the whole number procedure that was previously learned was applied. Having only experienced the calculation with whole numbers, the child is aware only of the procedure of aligning numbers on the right. Furthermore, the child has learned the procedure of right alignment through his or her experience of learning whole number calculation using vertical notation.

The diagram on the next page illustrates the process of the extension of the application of the whole number procedure. With regard to the introduction of whole numbers in situation I, the procedure for aligning decimals matches the meaning of place-value (arrow A). When children become accustomed to this procedure, they forget the meaning of place-value and become proficient in quickly aligning to the right (II). In the domain of whole numbers, the meaning of place-value is not contradicted even if numbers are aligned to the right (arrow B). However, when children apply this procedure to decimal numbers (III), it contradicts the meaning of place-value as shown in 1) (arrow C). Therefore, when children are faced with an instance when the procedure does not apply, they become aware of the gap and must once again return to the meaning of place-value. Then, they apply the procedure to both whole numbers and decimal numbers, and they become aware of the procedure of aligning decimal numbers as a procedure in accordance with the meaning of place-value.

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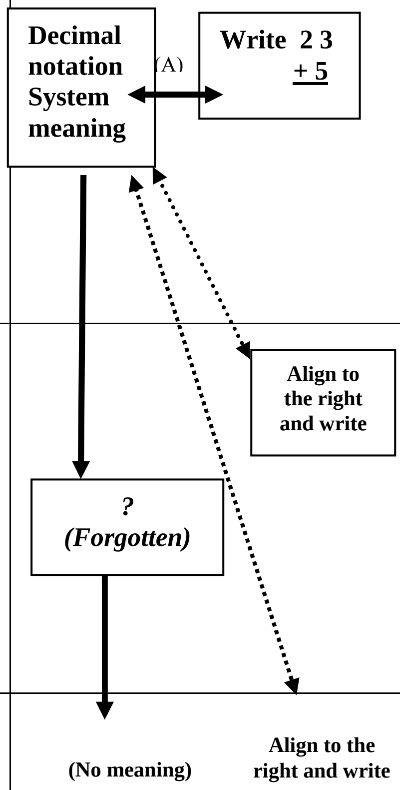
[3] Extension (extending or expanding situation) is a basic principle of Japanese curriculum and textbook sequence in mathematics. Thus, over-generalization by students can be anticipated by the teacher. The examples here may not be particularly special even for those in other countries because the extension is normal sequence in school mathematics without axiomatic mathematics at the age of

New Math.

Situation	Meaning Procedure	Explanation	Appropriateness
I Introduction of calculation in vertical notation using whole numbers		The meaning of a decimal notation system is based on the procedure of keeping decimal points in alignment. (The meaning and procedure match)	**Appropriate**
II Becoming proficient in whole numbers		When children become proficient, they no longer need to think about the reason they follow that procedure. As a result, the procedure is simplified from the alignment of the decimal points to one of right-side alignment.	**Valid**
III Application of decimal numbers		The procedure for whole numbers is generalized for decimal numbers	**Inappropriate**

Obviously, many children solve decimal number calculations using vertical notation through an understanding of the meaning of place-value. Thus the number of children who resort to the right-side alignment procedure is small. From the perspective of meaning and procedure, however, the way in which gaps in meaning and procedure occur tells us that there is a necessity in the teaching process to separate meaning and procedure into the following three categories. Children's levels of comprehension are by no means uniform in the process of learning. Comprehension develops differently in each child. While there are children who are no longer aware of meaning because they have become accustomed to applying quick and easy-to-use procedures, there are also children who are aware of meaning and use it as a basis for the procedures. Because the conditions vary, a diverse range of ideas involving previously learned knowledge appears in situations (extending situations) (III) when easy-to-use procedures do not work.

I) Deepening meaning: No appearance of gaps between meaning and procedure
"It goes well!"
II) Gaining an easy-to-use procedure from the meaning: Gaps are unrecognizable.
"It goes well!!"
Children become accustomed to easy-to-use procedures that work and many of them become unable to recall the meaning.
III) Situation where easy-to-use procedures do not work: Awareness of gaps
"What?"

The problems considered in Mr. Masaki's class and in exercise 2, the practice of expressing a number with one denomination in a form that uses multiple denominations, are examples of extending situations (expansion). In an extending situation, the procedures and meanings that have been established will not work, which means that they will need to be reconstructed. Taking the above decimal number calculation in vertical notation as an example, the meaning of place-value works, but the right-side alignment procedure needs to be revised. Accordingly, the meaning of place-value needs to be reviewed, and the procedures used need to be revised to ones that align the positioning of the decimals in accordance with proper place-value notation. In short, as an educational guide, category III can also be described as follows:

III') Reviewing of meaning and revision of procedure: Elimination of gaps

2-3. Diverse ideas can be classified by meaning and procedure

Up to this point, we have focused on the most extreme over-generalized ideas (misconceptions) to indicate the occurrence and elimination (bridging) of gaps between meanings and procedures. Naturally, in actual classes a diverse range of ideas will surface, including correct and wrong answers. In order to plan developmental discussions, it is necessary to anticipate the type of diverse ideas that will most likely appear. Here, let us treat the children's ideas as observations. For example, at the Sapporo City Public Konan Elementary School, Hideaki Suzuki's 5th grade class looks at division involving numbers with 0 in the end places. This class, as was the case with Masaki's class, first confirms previously learned knowledge of division when there is no remainder (task 1) and then moves on to the target content, which has yet to be learned: division when there is a remainder (task 2). The objectives of this class can be confirmed in the following discussion showing the flow of the class lesson (See next page).

Task 1. Known problem to confirm a previously learned procedure and the meaning it is based on: Previously learned task.

When children who have knowledge of basic division work out the answer to 1600÷400, the following is reviewed:

```
        4
400 )1600
     1600
        0
```

A. Take away 00 and calculate: procedure
B. Explain A as a unit of 100 (bundle): meaning
C. Substitute A for a 100 yen coin and explain: meaning

Task 2. Unknown problem that seeks an application or expansion of the previously learned meaning and procedure: Target task.

The target problem presented is 1900÷400, which presents a problem for some children and not for others as to how to deal with the remainder. As a result, the following ideas appear.

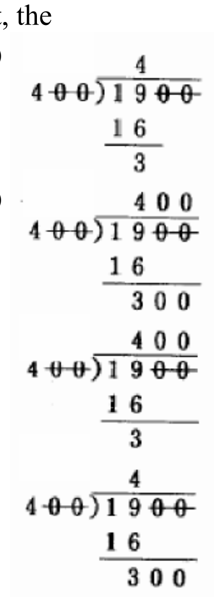
a) Answer to the question using a procedure in which the meaning is lost.
Apply A and make the remainder 3. Because the meaning is lost, the children do not question the remainder of 3. (Half of the class)

b) Answer to the question when procedures have ambiguous meanings.
Using A and B, the remainder was revised to 300. However, because the meaning was ambiguous, it was changed to 400. (Several students)

c) Answer to the question when the procedure is ambiguous.
A was used, but here a different procedure was selected by mistake. No students question the quotient 400. (Very few students)

d) Answer to the question that confirms procedural meanings.
Using A, an explanation of the quotient and remainders from the meaning of B and C.

a)
b)
c)
d)



Why do answers differ?

Where did you get lost? What did you have a problem with? A reminder of conflict through solving an exercise using your own ability.

By reviewing the solution process, the basic meaning is reconfirmed and the procedure for dealing with remainders is learned.

First, the children grapple with Task 1, which they have learned before. The teacher links this task directly to Task 2 in the target content of the class, keeping the children's solutions in mind. This is done by asking the children to confirm the procedure for the division using vertical notation, and asks them why it is not a problem to do this (meaning). Simultaneously, the teacher makes sure the children are able to explain both procedure and meaning. Following that, the children tackle target Task 2, which requires them to deal with remainders. In Task 2, a variety of ideas (a-d) appear among children who are doing the work without complete knowledge of the meaning, and among children who are confirming the meaning while working on the task.

Situation: confirming what they have already learned "It goes well": **Sense of Efficacy**

Mutual confirmation of meaning and procedure. Even if gaps in meaning and procedure exist, they do not appear here.

Situation: different from what they have learned before—Conflict

What?: **the unknown** due to an awareness of the gap with what they have already learned.

Some students experience such gaps in meaning and procedure whilst some do not.

What?: **Surprise at the difference in ideas** with other students and reflection on one's own ideas. Developmental discussion that correctly redefines meaning and procedure.

Acquisition of a sense of achievement, appreciation, by overcoming conflict and proceeding through to understanding

The objective this time is to have a developmental discussion regarding the place-value of the remainder being adjusted to the place-value of the dividend.

Here, it is important to have readers understand that the above approach is fixed in the class. It is worthwhile noting that even if meanings and procedures are previously confirmed, there is a diverse range of ways to process and implement that comprehension. As such, a variety of ideas appear. The starting point in the creation of diverse ideas lies in ways to process and utilize individually.

When categorizing the variety of ideas above (a-d) by meaning and procedure, the following category types can be identified. These are developed with reference to the extension task that followed the known problem used to confirm previously learned knowledge.

Type 1. Solutions reached through the use of procedures without (or regardless of) meaning: Prioritize procedure without meaning.

This is the above-mentioned idea a). It refers to an idea reached through consideration without much attention to meaning, even though the correct procedure (calculation) is applied. There are children who immediately change their ideas by recalling the meaning after having been asked to explain or after listening to other children's ideas. However, most children substitute meaning with procedure and when they are asked for an explanation they usually reply by describing their procedure, saying "I did this, then I did that." Prioritizing the procedure means that the children do not give careful consideration to the meaning; rather they tend to use quick procedures.

(In the case of an already known task, and if we apply the correct procedure, the answer must be appropriate, but now we are discussing the case of the extension task.)

Type 2. Solution reached through the use of procedures with meaning: Prioritize procedure with confused or ambiguous meaning.

This type is composed of ideas b) and c). These students have the intention of confirming the meaning of the calculation procedure, but their idea includes their own semantic interpretation. Therefore, when getting to the core of their idea, it is found that their idea is one that contradicts the meaning and procedure they have previously learned. As a result, there are many instances in which their idea brings about confusion and unease.

Type 3. Solution reached through the use of procedures backed by meaning: Secure procedure and meaning.

As shown in d), when a solution reflects the appropriate meaning and has been learned as a procedure, there are no contradictions between procedure and meaning.

Usually, when people are faced with a task they are unfamiliar with, the first thing they do is to test existing quick-to-use procedures in which they are proficient. This is what is referred to as the 'prioritize procedure' situation. If children believe in the situation that they got appropriate answere without considering meaning, then they are categorized as Type 1: 'prioritize procedure without (or regardless of) meaning.' In actual fact, there are many children who react to an unfamiliar task by prioritizing procedure without giving any careful thought to meaning. If children further investigate meaning when asked if the procedure they chose to implement is appropriate, and they show confusion and concern, they are categorized as Type 2: 'prioritize procedure with confused or ambiguous meaning.' In contrast, a careful student who tackles a problem by always investigating the meaning and making sure there are no gaps will produce a result that has a secure procedure and meaning; they are categorized as Type 3.

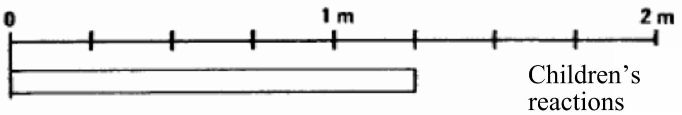
Although not shown in the above example, other ideas such as the following are also identified.

Type 4. Solutions through meaning only: Prioritize meaning without procedure (or confused procedure).
This type is seen when the procedure cannot be used appropriately or the student is not yet proficient in its use. Consequently, the solution is gained through thinking mainly about the meaning. As an example, consider the case where a student cannot calculate 1900 ÷ 400, but can answer if asked to solve the problem: "You have 1900 yen. You buy as many 400 yen pencil cases as you can. So..."
Type 5. Inability to find a solution due to insufficient meaning and procedure: No meaning or procedure.

It is particularly important for teachers to keep in mind Type 5 children who are unable to solve a problem. In the case of Type 4 children, they can give many possible reactions in class, but in many cases there is no result when it comes to formal tests. In the case of 1st and 2nd graders, many Type 4 children give reasonable answers if they have a good understanding of the meaning, but children in higher grades will meet difficulties. When elementary school children reach the 5th and 6th grades, and even more so when they enter junior high school, there is an increase in textbook and course materials that require the procedurization of meaning. If children do not have procedure, it is impossible to develop the meaning entailed by the procedure. So it is very important to be aware that some children in Type 4 will move into Type 5 without proficiency of procedure.

Here we would like readers to tackle the following problem regarding the meaning and procedural knowledge possessed by children from Katsuro Tejima's class.

Exercise 3 The following is used in the introduction of fractional numbers for 4^{th} graders. When asked to answer using fractional numbers for the length of a piece of tape, children's responses fall into one of three different types. Please explain what the children were thinking.



Children's reactions

① $\frac{5}{8}$m
② $\frac{5}{4}$m
③ 1m and 1/4m

Answers to Exercise 3

As previously taught in the third grade, a fraction is interpreted as a number of parts of the equal (even) divisions of a whole, and in the case of the fraction of a quantity, "2/3 m is the same as two parts of three equal divisions of 1 m". Fractions of one meter are learned only in the context of measurements of less than 1 m. This previously learned procedure tells children always to divide the whole number evenly and uses contexts in which the numerator never exceeds the denominator. The children's thinking may then be characterized as follows.

1. 5/8 m: the procedure was applied by making 2 m as one unit. This method is consistent with the procedure already learned; however these children did not recognize the contradiction inherent in obtaining a value less than 1. Accordingly, it illustrates Type 1: ‘prioritize procedure without meaning.’
2. 5/4 m: this answer was quickly found using the assumption that if there were three parts, each of which was 1/4 m, the total length would be 3/4 m, so that if there are five parts, the length should be 5/4 m (generalization of procedure). This contradicts the meaning and procedure children were previously taught, in which a numerator is smaller than denominator. Children who felt uneasy in this instance would be classified as Type 2: ‘prioritize procedure with confused meaning.’ Children who used the diagram to establish that 3 parts of 1/4 m becomes 3/4 m and so 5 parts becomes 5/4 m (meaning), but were then confused as to whether they could write that way because they had previously learned that the numerator cannot exceed the denominator (procedure), would be classified as Type 4: ‘prioritize meaning without procedure (or confused procedure).’
3. This answer shows that the children regarded the length as 1 m together with a further 1/4 m, obtained by subtracting 1 m from the total. As there is no discrepancy with what was previously learned, these children would be classified as Type 3: ‘secure procedure and meaning.’

This book focuses on lesson planning by teachers, and as previously mentioned, teachers ought to decide what the meaning and procedure are in their class material, and should provide appropriate educational guidance in accord with their teaching plan. It should be noted however that even when children are classified as the same type, their actual understanding, their thought processes and the ways they deduce meaning and procedure, may differ depending on the individual child and the situation.

Before each lesson, it is necessary for teachers to prepare teaching material and plan the lesson on the basis of the required curriculum sequence. In aiming to support lesson planning, this book has identified the above-mentioned types as part of the teaching material research carried out by the teacher. The teacher will be able to prepare the following in accord with the categorization by types: anticipate what kind of ideas will emerge from children based on what they have previously learned; plan well-devised instructional content for the class based on these diverse ideas; and create ways of facilitating the instruction so that children are able to recognize what they do not understand and are then led to experience the joy of understanding. By anticipating children’s ideas and the causes of possible confusion, teachers will be able to envisage beforehand how they should develop their explanations and discussions. The categorized types provided are for the teacher to use in order to plan lessons for conceptual development, based on what the children have previously learned, using extending examples or situations.

3. Planning for a Lesson with Developmental Discussion and Diverse Ideas

This section will incorporate what has been covered in previous sections and will demonstrate how to implement the wide range of ideas children create and show how to run a developmental discussion (dialectic) in the lesson. As already mentioned, the developmental discussion is planned for special occasions during the teaching sequence. If the curriculum or textbook sequence includes extending mathematical ideas, we can expect contradictions to inevitably occur. In the problem solving approach, we aim to develop mathematical communication as well as mathematical conceptual development. Thus, in this book, we are quite positive in promoting such contradictions as objects for discussion in the mathematics classroom.

3-1. Instruction planning in which a wide range of ideas appears by taking advantage of knowledge gaps

Here, the 'third grade decimals' lesson conducted by Junko Furumoto (Sapporo Midorigaoka Elementary School) will be used as an example. When teaching fourth grade lessons on decimals, it is known that children tend to over-generalize when they try to express a number with one denomination in a form that uses multiple denominations, as shown previously in Exercise 2. Ms. Furumoto recognizes this over-generalization as a gap that appears due to an extension of the procedure children have developed for dealing with numbers with only one decimal place to numbers with two decimal places. Accordingly, she has created the following lesson plan to take advantage of this gap and so add depth to her lesson on decimals.

1st class: In what situations are decimal numbers used? The existence of decimal numbers.

2nd class: How much juice is there? The need for decimal numbers (meaning).
1/10 dL = 0.1 dL: decimal numbers are used to express amounts smaller than one unit (meaning)

3rd class: Let's make a numeric line based on 0.1: the size of decimal numbers

It goes well!
The meaning and procedure match.

4th class: Let's get decimal numbers to introduce themselves: practice with large/small numbers and amount (meaning and procedure).
"I am 2.8. I am a number made up of two 1s and eight 0.1s."

5th class: How much is 3.7 cm or 1.5 L: practice in use of single and multiple denomination numbers. Re-expressing single and multiple denomination numbers (meaning and procedure).

It goes well!!
Procedurization, loss of meaning, or no loss of meaning.

6th class: There are two pieces of string: one is 4.2 m and the other is 4 m 10 cm.
Which one is longer?

What?
The occurrence of gaps.

The first five lessons, each of which is one hour long, are designed to deepen the children's understanding of the meaning of the first decimal place. In particular, the fourth and fifth hours focus on procedural proficiency (form) in terms of semantic interpretation. Up to this point, the method of instruction is standard. The sixth lesson is planned to make children wonder "What?" A diverse range of ideas appears as some children try to apply quick, easy-to-use procedures while others consider the problem using their understanding of decimal numbers, based on the example that 0.1 equals 1/10 of 1. It is planned this way so that conflict will occur. Furthermore, this conflict is used to get children to re-evaluate the meaning of place-value, including those children that did not have an accurate understanding of the meaning of decimals in the first instance.

The sixth class unfolds as follows:

- Preconception: It's 4.2 m! It's 4 m 10 cm!

How should I compare them?

- The units are different, so if I don't align them, I won't be able to compare them.

> What should you do so that you can clearly find out which

- For children who can't solve this problem by themselves, the teacher makes them realize
that they should use diagrams or the numeric value line they have previously learned.

a) 4.2 m = 4 m 20 cm, so... b) 4.2 m = 4 m 2 cm, so... (majority of the students)

c) 4 m 10 cm = 4.1 m, so... d) 4 m 10 cm = 4.10 m so...

- **Conflict:** a) vs. b), c) vs. d). Is 0.1 m 10 cm or 1 cm?
- **Returning to the meaning:** By converting the units to meters

(Using diagrams and number lines) 10 cm is 1/10 of 1 m, so it is 0.1 m
4 m 10 cm = 4.1 m < 4.2 m
By converting the units to cm
0.1 M times 10 equals 1 m, so it is 10 cm
4.2 m = 420 cm > 410 cm = 4 m 10 cm

> If the units are different, then compare them by converting them (procedure)

- Reaching understanding

A wide range of ideas appear in answers a to d. Children chose answers a and c based on the meanings they had learned up to the fifth class: "0.1 m is 1/10 of 1 m" (classify as Type 3: 'secure procedure and meaning'). Answer b may be the result of the quick procedure in the fifth lesson, which doesn't work (classify as Type 1: 'prioritize procedure without meaning'). Answer d may be an example of Type 2: 'prioritize procedure with confused or ambiguous meaning' if the children are confused as to why a contradictory expression that they do not understand appears. This is due to the fact that should they consider the quick procedure 4.2 m = 4 m + 2 cm to be correct and actually write 4.2 m = 4 m 2 cm, they will also necessarily write

4 m + 10 cm for 4.10 m. A similar case is where children wrote 4.10, because 1/10 of 1 m is 10 cm. If the children are confused as to whether they can write 0 in the second decimal position, then they should be classified as Type 4: 'prioritize meaning without procedure (or confused procedure).'

After the gap in ideas has been confirmed[4], the class moves on to encouraging children who chose answer d (with a question about 4 m 10 cm being 4.10 m if 4.2 m = 4 m 2 cm), to consider the problem in the context of answer b, in order to return to the meaning of decimals they had previously learned, which is that 0.1 = 1/10 of 1. Through discussion, the quick procedure is revised and the procedure for converting the units becomes clear. Further, children's understanding of the meaning of decimals, which observes a place-value of numbers, such as 10 cm = 0.1 m, is deepened.

It is worth noting that even though the first five hours of lessons have placed heavy emphasis on amounts and meaning through the use of specific examples and number lines, a large number of children will choose answer b. As previously mentioned, when adults learn a quick procedure, they will try to use that procedure in the first instance. Children are no different. When children become aware of easy-to-use procedures, many children are unable to recognize the semantics of the pre-requisite 'if...' of the procedure (in the 'if..., then...' structure). Ms. Furumoto's children would not have acquired even the easy-to-use procedures sufficiently without attending the sixth class. Accordingly, the aim of the sixth class is to deepen children's knowledge regarding procedures that convert units and the meaning of place-value in decimal numbers by continuing to detect insufficient understanding and then revising the meaning.

The diagram below shows a summary of the sub-unit construction mentioned above, focusing on meaning and procedure.

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[4] The difficulty in understanding other's ideas is that each of them is deduced from reasoning based on the different presuppositions depending on different understanding. In order to understand each other, it is necessary to reason based on others' presuppositions or to identify the necessary presuppositions from which may be deduced other's ideas. This point is focused on the third book (Isoda&Kishimoto 2005).

I) Constructing meanings

1st – 5th class: Matching meaning and procedure. No gaps become apparent.

Specific amounts, number lines and diagrams are used to learn that 10x0. 1 amounts to 1 (meaning).

II) Constructing easy-to-use procedures with meanings as the base

Part of the 4th class: the following quick rewording is taught, “2.3 is made up of two 1s, and three 0.1s.”

Part of the 5th class: Becoming proficient in procedure. Some students begin to lose the meaning of the procedure.

5.3 cm = 5 cm 3 mm, 2.7 L = 2L 7dL can be re-expressed quickly.

III) The situation of easy-to-use procedures not working: Extending the situation
The meaning is reviewed and the procedure is revised

6th class: the gap is exposed between the solution brought about from the procedure whose meaning has been lost and the solution that reflects the meaning. Then conflict occurs, leading to a review of the meaning of the procedure and a revision of the procedure itself. Through this, a new understanding is achieved.

Children who apply the procedure from the 5th class.
4.2 m = 4 m 2 cm
Meaning loss from the 1st to the 5th classes.

Children who solve the problem using the meaning learned from the 1st to the 5th classes.
4.2 m = 4 m 20 cm
4 m 10 cm = 4.1 m

The meaning of a place-value in decimal numbers is reviewed and acknowledged. Then the procedure for re-expression of numbers in different denominations is revised.

* The discussion structure of section III includes the Hegelian meaning of the dialectic process through sublation. Here, Other’s different ideas are functioning antithesis. We will discuss this later.

The climax of the sub-unit construction is section III. What is the process of reaching section III? First, in section I, procedures are learned while keeping meaning in mind. In section II, an easy-to-use procedure is acquired. As children become proficient in this procedure, some of them lose the need to consider meaning. In section III, they are faced with instances in which the easy-to-use procedure does not work.

At the stage of solving problems by themselves before whole classroom discussion, each child may become confused because the easy-to-use procedures do not always work. When they participate in developmental discussions, conflict arises regarding the difference in ideas held by other children. By experiencing that conflict, the meaning as a basis for supporting the procedure, which many children lost in section II, is once again recognized with a higher form of generality, and then the procedure is revised.

The following describes the process of sub-unit construction in more general terms.

The development of sub-units in which divers ideas appear

It goes well!

I) Deepening the meaning: when meaning and procedure match

Here, meaning is deepened by being matched to procedure.

II) Constructing easy-to-use procedures based on meaning

Here, the 'procedurization' of meaning is developed and students become proficient in easy-to-use procedures. At that time, some students fail to remember the original meaning. Even in such cases, however, the procedure will continue to result in a correct answer and no gaps in understanding will become apparent. Therefore, students experience no confusion.

It goes well!

III) A situation when easy-to-use procedures do not work; a review of meaning and a revision of procedure

Gaps in understanding are exposed when some students use a procedure without keeping meaning in mind and others correctly solve the problem because they remain aware of the importance of the meaning of a calculation. This causes conflict, and after reviewing meaning and procedure, a new level of understanding is reached.

Extending situation: What?

As these cases show, due to the fact that the loss of meaning that accompanies procedurization occurs slowly, it is not always possible to differentiate between sections I and II. The major question is how to work towards the climax in section III. In other words, how do teachers teach in order to enable children to overcome the conflict? Looking back on the examples, the following two points, A) and B) must be necessary conditions.

A) Posing tasks which, with poor understanding, will produce different answers.

Tasks should be presented in such a way that there will be a conflict between children who forget or do not care about meaning in acquiring the easy-to-use procedure in section II and children who do keep meaning in mind. In order to do this, tasks must be presented in which children will get stuck or there will be contradictions when easy-to-use procedures are applied in extension situations without due care for meaning. These children may develop their own ideas which should be changed, or they will need to reconsider the meaning.

B) Preparation of meaning that will function as the ground for developmental discussion (a dialectic) and a basis for understanding

For overcoming conflict due to difference in ideas (Hegelian sublation), it is necessary for the children to understand meaning (section I) because this meaning can be used as the basis for the developmental discussion.

In fact, because conflict arises by posing suitable tasks (see part A), or in other words, children encounter results completely different from their own, they are able to ask "What?" or " Why?" This allows them to reflect on their own ideas and take part in developmental discussions as they compare their ideas with those of others. Additionally, the mutual result from this confrontational developmental discussion makes the children produce a response to explain why they arrived at different answers. In the developmental discussion, part B is also necessary. The reason for this is that if the children cannot understand others, or if they cannot accept other's ideas, or if they cannot reproduce other's ideas, their discussion has no common ground as a basis on which to argue and talk at different purposes. If they have a basis for discussion, they can reflect on what others are saying.

When children actually ask each other "Why?", those children who resorted to the easy-to-use procedure (classified as Type 1: 'prioritize procedure without meaning') can do nothing but answer: "Last time 1.5 L was 1L and 5 dL, right? So I did it the same way for 4 m 2 cm," or "You do not make 4 m 10 cm into 4.10 m (in other words, "You do not write it that way") right?" Next, children who correctly applied the meaning to the solution began to talk about the basis (meaning) of the procedure by saying "0.1 m is 1/10 of 1 m, right?" By working out the difference in the meaning of place-value for a dL from the previous time and the relationship between meters and centimeters, the meaning becomes clear. The children who only applied the easy-to-use procedure, and were not conscious of the meaning, now become able to reproduce the correct results. Children who are satisfied with the meaning as discussed are able to revise their own ideas.

3-2. Planning a one-hour class with confirmation of previously learned tasks to reinforce children's knowledge and target tasks

The method indicated for sub-unit construction is also useful for planning a one-hour class. That is, as previously discussed, it demonstrates how to structure a lesson that involves previously learned and target tasks. Here, we will explain Katsuro Tejima's (Joetsu University of Education) introduction to fractions for fourth graders by way of meaning and procedure, and we will show the flow of his lesson structure

(Ref: “Kazu-e-no Kankaku Wo Sodateru Shido,” Elementary School, University of Tsukuba).[5]

Third grade
3/4 Three parts of four equal divisions of a whole.
3/4 m This is composed of three parts of four equal divisions of 1 m.

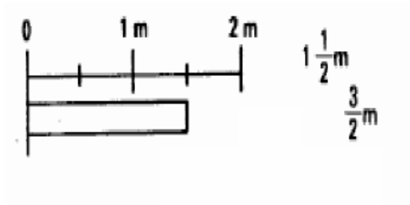
Extending ↘ ↘ ↘ ↘ ↘ **Extending**

Fourth grade
1 and 1/4 m... This is 1 m and one part of four equal divisions of the other 1 m. 5/4 m This is five parts of four equal divisions of 1 m???

3/4.....Three times 1/4 unit
(Three times one quarter)
5/4.....Five times 1/4 unit
(Five times one quarter)

5/4 m.....Five times 1/4 unit
(Five times one quarter)
→ Collection of unit fraction

First, Tejima revises the meaning of fraction learned in third grade, before improper fractions are introduced in fourth grade (see diagram above). Because “five parts of four divisions of 1 m” makes no sense, it is necessary to teach children about the way of looking at improper fractions as a collection of unit fractions. Also, he tries to utilize the gap between meaning and procedure that occurs in the children’s thinking. In the third grade, even when children study the meaning of “3/4 m is 3 parts of four equal divisions of 1 m”, there are children who learn it as the procedure: “if it is 3/4 m, then take three of the four equal divisions of the whole” because they only learn in the case of equal divisions of the whole. As a result of applying the procedure, 2 m is seen as the whole and the answer is given as 3/4 m.



[5] In Japanese elementary mathematics education, a fraction is first introduced via a situation such as dividing up a pizza or a cake. In this context, it is explained by the part-whole relationship (fraction without denominator). Second, a fraction such as 1/3 m is introduced (fraction with denominator). In this context, the meaning of a fraction is extended from the part-whole relationship to the number line with the idea of a quantity. Thus the improper fraction 4/3 means four lots (four times) one third (one third as a unit fraction). Later, a fraction is recognized as the result of division (for example, the special case of decimal fractions). Finally, a fraction is recognized and interpreted as a ratio. The lesson by Masaki was given based on past curriculum standards (1980). In grade 3, a fraction is introduced as a relation between parts and a whole. Mixed fraction, Improper fraction, Proper fraction, and Unit fraction are taught in 4th grade. The sequence changed a little in 1999 standards.

He used the following structure for a single lesson that incorporates previously learned tasks and target tasks. The aim of the lesson is to bridge the gaps between meaning and procedure that children hold and to clarify misconceptions about the meaning of fractions.

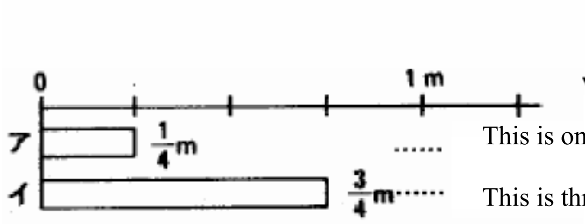
Previously learned task 1: The teacher shows the children a 1 m long piece of tape and divides it into four parts in front of them. He asks them: “How long is each part?”

C1: 25 cm, C2: 0.25 cm, C3: 4/100 m, C4: 1/4 m

Previously learned task 2: After confirming that the length is expressed as the fraction 1/4 m, the teacher says: “Today, let’s express the length of this tape in fractions.” He then cuts the tape into two pieces: 1/4 m and 3/4 m. As shown below, the teacher then asks: “How can we express lengths A and B in words? First, let’s think of A as 1/4 m.”

C5 This is one part of four divisions.

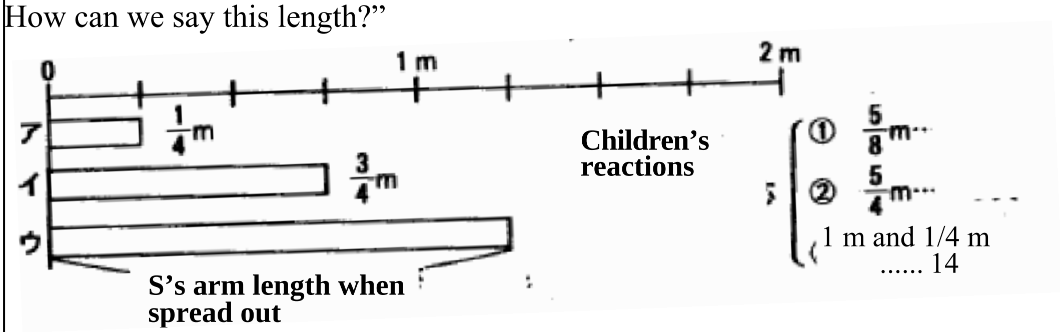
C6 This is one part of four divided parts from 1 m.



This is one part of four evenly divided parts from 1 m.

This is three parts of four evenly divided parts from 1 m.

Target task: Next, the teacher takes out a piece of tape measuring 125 cm. He then says: “T, the length of this tape has a connection to the human body. What do you think it is?” Following this, the teacher develops the discussion by saying: “C, the length of both arms spread out. It is an actual fact.” He then says to the children, as indicated in the diagram below, “When S spreads his arms out, the length is over 1m. How can we say this length?”



The developmental discussion unfolds via a debate about tasks 1 and 2.

C9 I think 5/4 m is strange.

C1 0 It's five parts of the four divisions of 1 m.

C (to C9) That's right./ I disagree.

C11 I disagree. If you take the 1 m away, 1/4 m is left. 1 m equals 4/4 m, so if you put them together, it's 5/4 m.

C1 3 5/4 m is strange because even though 1 m was split into 4 parts, the numerator is bigger than the denominator.

C14 There are one, two, three, four, five lots of 1/4 meters, so it's 5/4 m.

C1 5 If it were 5/8 m, then it would mean it was the fifth part of eight evenly

Summary

If it is 5/8 of 2 m, then that is correct.

If 5/8 m is written with 'm', then it becomes smaller than 1 m, which is strange. It is five times the 1/4 m tape length, so 5/4 m is ok.

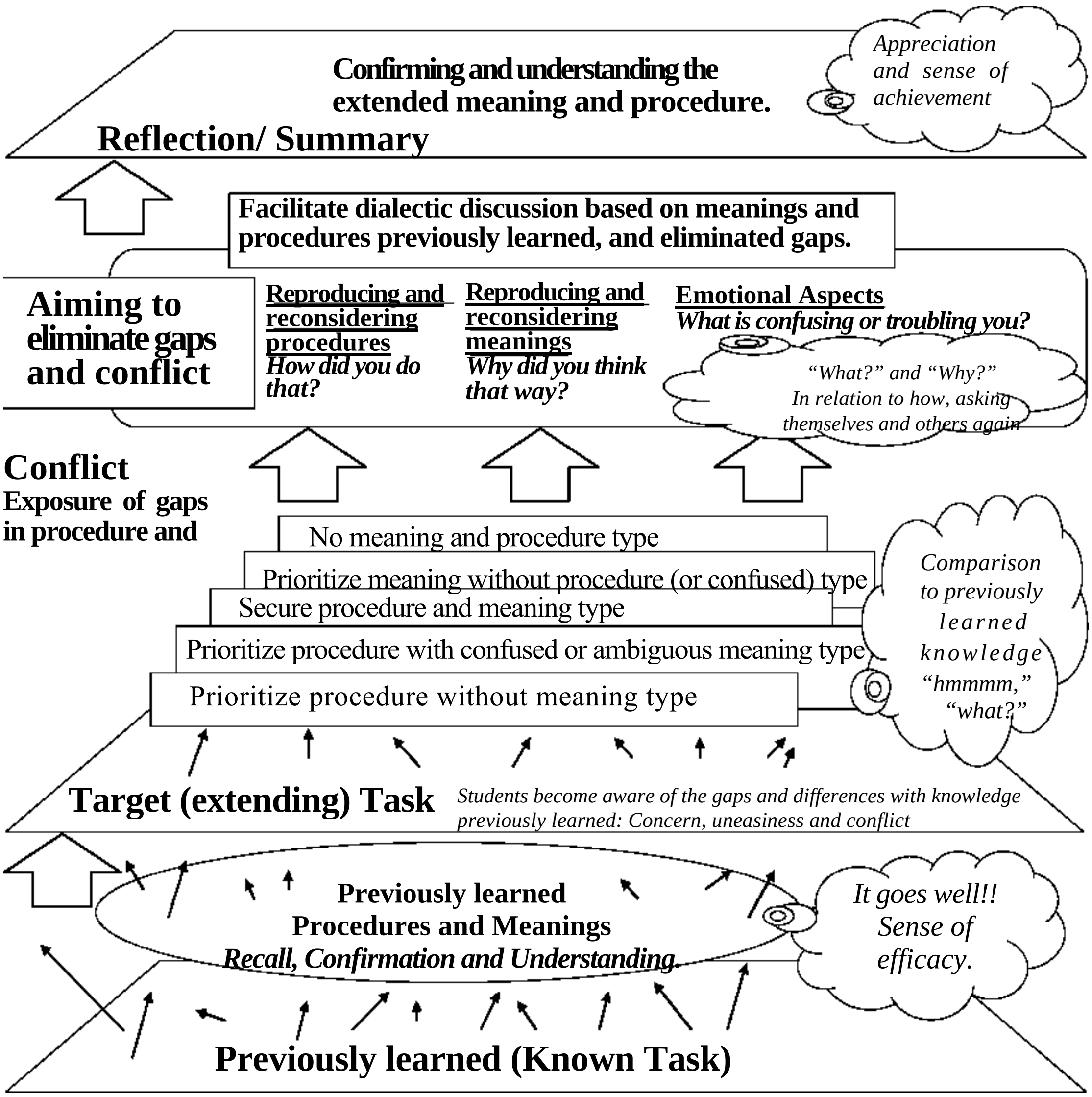
The above is an overview by Tejima. What would have happened if the teacher had begun the class by skipping the review of previously learned material and immediately used the target task? Since the target task is an extension of the previous material, a wide variety of ideas would appear. The developmental discussion would have gone out of control and continued in the same way if children had not shared the grounded meaning of Task 1 (see, Isoda, 1993).

He knows that many children will come up with the answer 5/8 m before he plans the lesson[6]. The goal of this class is to make children aware of a new meaning of multiples of a unit fraction, so that this may serve as a basis for a procedure known as improper fraction representation, which will be covered in the next lesson. To that goal, it is necessary to emphasize to children the idea of aggregating a number of fractions of unit 1/4 m. (Children do not know about a fraction as a unit, or as a number on the number line). At the same time, it is also necessary to revise the misunderstanding of 5/8 m, which comes about from thinking of fractions as equal parts of a whole. In order to revise this idea, he reminds children to consider the length in Task 1 and asks the children if they can confirm that 25 cm = 0.25 m = 1/4 m. In Task 2, he reviews the definition of fractions, confirms it and tests it in the

6 In Japan, the results of lesson studies such as children's ideas in the context of teaching on curriculum sequence have been well shared through teachers' guidebooks and journals. Thus, teachers can expect children's response before the lesson.

target task by placing the 1/4 m and 3/4 m in the tape diagram on a number line in increasing order By creating this contextual flow, it is easy to become aware of “how many 1/4 m parts” there are, such as in the answer 5/4 m. Further, the idea of 5/8, which was obtained without meaning, is “5 parts of 8 equal divisions of 1 m.” This was obtained by applying the previously learned definition of fractional numbers. Children will realize that 5/8 m is smaller than 1 m. Here, counter-examples are effective: “5/8 m is smaller than 3/4 m, so it’s not right.” The developmental discussion was successful, as the meaning and procedure that form the basis of discussion had been confirmed in Task 1 and Task 2 before considering the meaning and procedure in target Task 3.

In conclusion, the lessons of Masaki on parallelism, Suzuki on division and Tejima on fractions can all be summarized as shown in the flow chart below.



In order to run a lesson to include such a flow, the following work (A-D) is necessary for its planning.

A) Investigate which stage of extension this class is at within the curriculum sequence, and what kind of changes are necessary regarding procedure and meaning to achieve the class goals.
B) Consider what types of target tasks are necessary to extend the material.
C) Anticipate what kind of reactions and gaps in meaning and procedure will appear when the children in the class tackle the target task, learned from previous situations.
D) Prepare tasks that review previous material to determine what needs to be covered in terms of meaning and procedure in order to perform the target task. This will also allow the creation of a basis for developmental discussion, which will examine what grounding of meaning is necessary for the elimination of gaps that appear during the target tasks.

If the lesson is developed as a part of a unit or subunit plan for teaching such as Furomoto's lessons on the decimal number, the first part of the lesson flow chart, that is, the previously learned task, can and often is put into the immediately preceding class in consideration of the above, and the lesson usually focuses on the remaining four parts: Target Task, Conflict, Eliminating Gap and Reflection.

3-3. Developmental discussion to eliminate (bridging) gaps

Upon reflection, developmental discussion takes place with the aim of eliminating conflict caused by gaps.

Developmental discussion (a dialectic) that eliminates gaps in diverse ideas

The reactions of children who are no longer aware of the meaning of the procedure	⟺ **Conflict**	The reactions of children who remain aware of the meaning of the procedure

Children who are aware of the meaning of the procedure and children who are not aware of it contradict each other. Here, the discussion develops based on the ideas and concerns of children who have an ambiguous understanding of meaning or procedure.

In order to eliminate contradictions and gaps, it is necessary for children to persuade other children to revise their ideas.

Considering what has been discussed so far, it is conceivable that developmental discussion will progress in the expected direction if the following two points are taken into account.

1) Consciously developing “Hmmm” and “Why?”

When children are solving new problems by themselves, they become concerned and uneasy and think “Is it ok to do it like this?” This concern and uneasiness are manifested in children’s feelings when they find gaps in the meanings and procedures of previously learned tasks. However, once the children have successfully answered the task question, they feel better and forget these types of feelings. If children lose the desire to eliminate concern and uneasiness from within themselves, they cannot understand the complex ideas of others. Moreover, they are unable to take note of the viewpoint of others and revise their own ideas by sharing their opinions with their classmates. Children who ‘prioritize procedure with confused or ambiguous meaning’ or ‘prioritize meaning with ambiguous procedure’ often display this type of concern and uneasiness. Therefore, the use of such concerns and uneasiness makes it easy to access the benefits of developmental discussion.

2) Sharing understandings of meanings which will serve as the basis for the developmental discussion

Mutual differences in procedures are exposed as gaps during the developmental discussion. In order to eliminate such gaps, children must talk about the meanings of the basis for each other’s procedures by asking: “Why did you think that way?” In addition, if they do not share or understand each other’s interpretation, they cannot revise their own procedures.

Using the above two points as a premise, the following two points can be shown as measures to set up and summarize developmental discussion.

a) Searching for a mutually recognized meaning to enable children to share a logical explanation as a base.

b) Using other’s ideas even when recognized as inappropriate and deducing contradictions.

It is fundamental for a developmental discussion to be planned with regard to point (a): It is necessary for mathematical explanation as a kind of mathematical proof. However, it is not easy for children to share the meanings. This is because it is difficult to respond when listening to another person’s comments. If children quarrel, a proper debate becomes hard to establish and those involved cannot break away from their own ideas and assertions. Here, the following teaching skills become necessary (see such as Kimiharu Sato, 1995).

- When different ideas are outlined, give children time to reconsider why they think their idea is appropriate, so that they can explain why they think that way.
  Example: Get children to write down their ideas regarding why they think that way.
- Develop the points of confusion and concern as points of discussion in order to organize them within the developmental discussion.
  Example: Ask children to comment on their points of confusion and concern.
- Organize the points of discussion so that arbitrary comments do not cause the developmental discussion to get out of hand.
  Example: "Try to say that again," "Hold on, I understand what he/she said," "That's good.
  Can someone rephrase it?" "Well, the points of discussion are on different levels now. Let me restate the problem."

Using these teaching techniques, the teacher encourages children to find meaning that everyone is satisfied with and ideas can be presented logically based on this meaning. In such a developmental discussion, point (b) above usually becomes necessary. In the first part of point (b), presuming that 'the other person is right' is a necessary condition for considering the other person's perspective. In other words, what is the premise used to enable children to reach such a result? In order to reach this result, children are required to determine what premises the other children are basing their ideas on. However, it is not an easy task to reproduce another person's ideas. In actual fact, when performing a task which exceeds the 'if' conditions of a procedure that works, it is not uncommon that more than half of the children misconceive the problem and use a procedure without any meaning. Among those children, some answer the way they do because they are unable to understand the reason for that meaning and seek to understand its basis. In that case, even if they listen to another person's explanation, they cannot agree with the other person's idea due to the fact that they are unable to understand what the other person is talking about, because they cannot understand the premise on which that person's idea is based. When this happens, first it is necessary to make the children aware that failing to take the premises into account will cause confusion. A persuasive technique is to suggest that the person temporarily accepts the other's idea even if it is very different from his/hers, continues to use the idea in another case, and then shows that it will contradict what they already learned before (the latter half of (b)). This is the Socratic dialectical method used since ancient Greek times, and is the origin of the reduction ad absurdum (reduction to absurdity) in terms of mathematics. In simply words, it is the production of a counterexample. If the other person does not understand it as a counterexample, it is not effective. Accordingly, the following section examines two methods that are effective in creating counterexamples.

a. Waiting Counter Example on (b) for (a): What if A's idea is correct?

Here is an example. Hidenori Tanaka, a teacher at Sapporo Municipal Ishiyama-Minami Elementary School, is teaching fifth graders addition of fractions with different denominators using the example 1/2 + 1/3. Some of the children give the answer 2/5. This answer shows a student in the 'prioritized procedure without meaning'. These children merely added the numerators and denominators of the fractions together, without understanding the meaning. Further, some children advocated the mistaken meaning by arguing (○●) + (○●●) = (○○●●●) ('prioritizing procedure with confused or ambiguous meaning'). For children who think this explanation is correct, it shows a lack of understanding of fractions, since it is impossible to add fractions together which are in different units. For this reason, even if the children were able to understand their classmate's explanation using a diagram, they would not understand why a classmate would say their own diagram explanation was wrong. What disproves their misguided understanding is the rebuttal, "So, have you ever added up denominators before?" According to this procedure, 1/2 + 1/2 = 2/4 = 1/2, and as the children see it, (○) + (○●) = (○○●●). Looking at it this way goes against what has been previously learned. Accordingly, this type of refutation, which is not a straight denial of that person's idea, uses their answer as an opportunity to critique their way of thinking, and is therefore quite convincing.

b. Facilitating awareness through application of tasks in different situations and examples

The excellent approach of asking "What if A's idea is correct?" is that it makes use of A's procedure without meaning. It includes the reasoning based on other's saying for trying to share the grand of discussion (a). In doing so, it focuses on the contradiction in procedure that the student has used rather than the meaning he or she does not understand. The use of A's procedure allows him or her to realize his or her own misconception of the procedure. This is the same method seen in Tejima's class.

However, there are also times when a contradiction needs to be indicated in new tasks in the case, a counter example is not clear for students or not given by students and a teacher does not show it.[7] Here, we present a following example of this method using a third grade fraction class run by Mikiko Iwabuchi, a teacher at Sapporo Municipal Kitasono Elementary School in Sapporo. In this example, a shift from fractions as equal parts of a whole to fractions as quantities on the number line (unit fractions) is planned.

In this planned lesson sequence (see next page), the meaning of fractions as equal parts of a whole is used as a basis for defining fractions as quantities in their own right. This definition creates a shift in meaning from "n parts of m equal divisions of the whole" to "n parts of m equal divisions of a unit quantity." Up until the second lesson, children have only studied fractions as equal parts of a whole, so there are various discrepancies in the semantic interpretation of the answer as 1/4 m in

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[7] If children well educated enabling to change the parameters on the problem by themselves and children have rich custom to explain their idea with the words 'for example', posing counter example by children is not rare case in elementary school classroom (See such as Tanaka 2001). Even if there is a child find the counter example against the answer, it is not always understandable for other children.

the third class. The students answers are wide ranging.[8] Debate arises among the children, and as expected, conflict is seen between those who chose answer B and those who chose answer C. In particular, as 1/4 m is read as '1 of 4 parts' m in Japanese, it is easy for the children to arrive at the idea that the number is four times the standard 1 m. As an idea to support C, one child claimed "it should be shorter than the original length" to make use of the meaning studied of fractions as equal parts of a whole. Another is the indication expressed in the comment: "If 1/4 m =1 m, you should say 1 m, otherwise it's strange." However, because the meaning of 1/4 m is undefined and discrepant, the children listening others will not be able to make sense of it. Therefore in the fourth class, the children are asked about the case of 1/2 m by the teacher. If B is correct, 1/2 m = 1 m and 1/4 m = 1m, and so you would have "1/2 m = 1/4 m," which again is strange, and a debate centering on "it should be shorter in the order of 1/2 m, 1/4 m, 1/10 m," would occur from the perspective of what was learned about fractions as equal parts of a whole. In other words, a conclusion that answer C is correct can be reached because the meaning and logic of fractions studied in the second class does not match answer B from the first class.

1st lesson: Halves... dividing equally... introduction of fraction as part-whole relationship using 1/2.

It goes well!

2nd lesson: "Let's make 1/4." Using fraction as parts of a whole. *It goes well!*

The teacher asks children to make a 1/4 size piece of colored paper and tape to send to their sister school, Astor Elementary, for its music festival.

3rd lesson: "Let's make 1/4 m." Introducing fraction as a quantity. *What?*

The teacher wants the children to cut a 1/4 m length of tape to send to their sister school's festival.

They must make sure the measurement is right.

A) The original size of the tape can be any size, so if the whole length is not given, it is not set. (2 children: 'Prioritize procedure with confused or ambiguous meaning.')

B) 4 m is divided evenly, each piece is 1 m. (16 children: 'Prioritize procedure with ambiguous or no meaning.')

C) One piece from 1 m is divided evenly (25 cm). (19 children: 'Secure procedure and meaning type.')

4th lesson: 'Let's make 1/2 m." Introducing fraction as a quantity (continued from the 3rd lesson).

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[8] Here, when the meaning matches the definition, it is classified as 'secured meaning; however, as this is at a stage before definition, it does not mean that others are misconceptions.

Based on the above discussion, the second chapter will show the practice of developmental discussion classes that lead to the creation of diverse ideas..

Notes & References in 1996 Japanese version.

From the viewpoint of academic research, the following is an explanation of the research path, its position in mathematics education, as well as the reference materials used in making this book.

In the early 1980s, it can be said that the theoretical framework for the problem solving approach, as it is now known in Japan, had already developed. In actual fact, the contents provided at that time, do not differ much from the research that had been done after constructivism became a significant issue for debate in the mid 1980s. Furthermore, as far as teaching practice is concerned, the level of lessons run by teachers using problem solving techniques in Japan ranks very highly, even from the perspective of constructivists. For example, Jere Confrey (vice-chairperson of International Group for the Psychology of Mathematics Education in 1995, when th book was written), a leader in the field of sublation of radical constructivism and social constructivism, has given a high evaluation of the idea as a constructivist approach in the lessons.

However, in the early 1980s and 1990s, there was a gap. For example, in the early 1980s, the discussion of diverse ideas was in terms of the diversity of correct ideas with open-ended problems. One factor that changed that trend was research about understanding. This chapter has been written to include the way to describe the phases of understanding – conceptual knowledge and procedural knowledge theory – as of the context of research on understanding, as well as to show the theoretical aspects of the problem-solving lesson and teaching practice of teachers from Sapporo.

The following papers act as a framework for this chapter.

Masami Isoda (the author), *"Katto to Nattoku wo Motomeru Mondai Kaiketsu Jugyo no Kozo,"* Riron to Jissen no Kai Chukan Hokokusho, 1991

I have studied much from the following researchers in order to acquire my theory:

Toshio Odaka & Koji Okamoto: *Chugakko Sugaku no Gakushu Kadai.* Toyokan Publishing Co., Ltd., 1982

Tadao Kaneko: *Sansu wo Tsukuridasu Kodomo*. Meijitoshoshuppan Corporation, 1985

Katsuhiko Shimizu: *Sugaku Gakushu ni Okeru Gainenteki Chishiki to Tetsuzukiteki Chishiki no Kanren ni Tsuite no Ichi-kosatsu*. Tsukuba Sugaku Kyoiku Kenkyu, 1989 (co-authored with Yasuhiro Suzuki)

Katsuro Tejima. *Sansuka, Mondai Kaiketsu no Jugyo*. Meijitoshoshuppan Corporation, 1985

J. Hiebert. *Conceptual and Procedural Knowledge: The Case of Mathematics*. LEA, 1986

Toshiakira Fujii. *Rikai to Ninchiteki Conflict ni tsuite no Ichi-kosatsu*. Report of Mathematical Education, 1985

The originality of this book lies in the following areas: applied a descriptive research method of children's understanding in psychology to lesson material and planning; and, applied the viability of knowledge on constructivism to the developing problem situations due to gaps in procedure and meaning that come about from extending and generalization on curriculum sequence.

James Hiebert, who is known as the conceptual and procedural knowledge theory, has appraised these applications.

Below are the references and contents that could not be included in the book although they too are worthy of use in this context.

Author's material:

Sansu Jugyo ni okeru Settoku no Ronri wo Saguru, Kyoka to Kodomo to Kotoba. Tokyo Shoseki Co., Ltd., 1993.

Miwa Tatsuro Sensei Taikan Kinen Ronbun Henshu-iinkai-hen. *Gakushu Katei ni Okeru Hyougen to Imi no Seisei ni Kansuru Ichi-kosatsu, Sugaku Kyouikugaku no Shinpo*. Toyokan Publishing Co., Ltd., 1993.

Sugaku Gakushu ni Okeru Kakucho no Ronri – Keishiki Fueki to Imi no Henyo ni Chakumoku Shite. Furuto Rei Sensei Kinen Ronbunshu Henshu-iinkai. Gakko Sugaku no Kaizen. Toyokan Publishing Co., Ltd., 1995

Mondai Kaiketsu no Shido. Shogakko Sansu Jissen Shido Zenshu 11 Kan, Nobuhiko Noda (Ed). *Mondai Kaiketsu no Noryoku wo Sodateru Shidou*. Nihon Kyouiku Tosho Center, 1995

Kimiharu Sato. Neriai wo Toshite Takameru Shingakuryoku. Kyouiku Kagaku, Sansuu Kyouiku September 1995 issue

In Japanese original version of this book, some words are used with special meanings even if in Japanese. For example, the phrase 'developmental discussion' has been used to describe the aim of restructuring meanings and procedures that children have through dialectical conversations with them. Furthermore, from the standpoints of 'if there is nothing extraordinary, then the idea cannot be truly tried or structured' and 'extending the concept cannot be done without the risk of over-generalization,' we replaced the word 'error (Ayamari in Japanese)' with 'over-generalized idea (Kari but read Ayamari in Japanese)'. This is in line with the meaning of misconception and at the same time is used in the background of an alternative framework on the theory of constructivism.

TEACHERS' MATHEMATICAL VALUES FOR DEVELOPING MATHEMATICAL THINKING TROUGH LESSON STUDY

Alan J. Bishop
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1. Mathematical thinking from a sociocultural perspective

Mathematical thinking sounds like an essentially psychological topic. It is just another branch of thinking, and therefore part of the psychological field of knowledge. However, we can never observe mathematical thinking - we can only observe what we assume to be its products, namely mathematical ideas and processes. We can also observe what conditions and contexts might have been responsible for the products of mathematical thinking, which brings us rather closer to the social context.

So what is the problem we are trying to consider here? In one sentence the major problem seems to be: "How can teachers help mathematical thinking to develop in their students?" A subsidiary problem is "How can research on values help with this?" Because of my research work in the field of mathematics education, I prefer to consider mathematical thinking not from a psychological perspective but from a socio-cultural perspective.

2. Three theoretical ideas

In trying to make research progress in solving the problem of helping mathematical thinking develop, I believe we need to consider carefully any theoretical perspectives which might assist us. I will present here three theoretical ideas which I have found helpful in my research and which I believe can shape our understanding of the problem and lead to potential pedagogical solutions. These 'solutions' can then be researched using the Lesson Study method – but more about that later.

2.1 Lancy's developmental theory of cognition

David Lancy (1983) is a cultural psychologist who, in his major cross-cultural study in Papua New Guinea, developed a new stage theory of cognition. It was based on Piaget's theories but he developed them from a socio-cultural perspective. He was doing his research in Papua New Guinea and through investigating cognition with students in PNG, he found that the theoretical developmental sequence of Piaget's stages were similar to, but not identical with, those Piaget found in his European-based research.

He found that Stage 1 was very similar to Piaget's sensory-motor and early concrete operational stages. He argued that this stage is where genetic programming has its major influence, and where socialisation is the key focus of communication. Many activities involving the child are completely similar across cultures.

He then argued that Stage 2, a later concrete operational stage, is where enculturation takes over from socialisation. As he says: “Stage 2 has much to do with culture and environment and less to do with genetics”, and he demonstrated that this is the stage where different cultures will emphasise different knowledge and ideas. Even in relation to mathematics (which is where ethnomathematics develops) this is the case..

The big development in Lancy’s theory from Piaget’s is seen in Stage 3 which concerns the meta-cognitive level. Lancy says: “In addition to developing cognitive and linguistic strategies, individuals acquire ‘theories’ of language and cognition.” Different cultural groups emphasise different ‘theories of knowledge’ and Piaget’s ‘formal operational’ stage is one such theory of knowledge emphasised in Western culture. Confucian Heritage Cultures emphasise other theories of knowledge. These theories of knowledge represent the ideals and values lying behind the actual language or symbols developed by a cultural group.

Thus it is in Stages 2 and 3 that values are inculcated in the individual learners. In a classic work by Kroebner and Kluckholm (1952) they strongly support this idea: “Values provide the only basis for the fully intellible comprehension of culture, because the actual organisation of all cultures is primarily in terms of their values” (p. 340).

Thus for our problem, the idea of mathematical thinking as a form of meta-cognition, affected by the cultural norms and values of the learner’s society, is helpful.

2.2 Billett’s (1998) analysis of the social genesis of knowledge.

But where do these norms, values and knowledge come from, and how can we think about them from a more educational perspective? Stephen Billett’s (1998) sociological work analyses and locates what he calls “the social genesis of knowledge” in 5 inter-relating levels:

Socio-historic knowledge factors affect the values underpinning decisions made by both institutions and teachers. It is knowledge coming from the history and culture of the society, and is value-laden knowledge.

Socio-cultural practice is defined by Billett as historically derived knowledge transformed by cultural needs, together with goals, techniques, and norms to guide practice. At the institutional level these are manifested by curricular decisions influenced by such factors as: (a) current institutional management philosophy with respect to educational and social values (in loco parentis); (b) State or national curricular frameworks and (c) the ethos of the mathematics faculty or teacher’s peer group.

The community of practice in the classroom is identified by Billett as particular socio-cultural practices shaped by a complex of circumstantial social factors (activity systems), and the norms and values which embody them. This community is influenced by (a) the teachers' goals with respect to and portrayal of pedagogical values, (b) students' goals and portrayal of learning values, and personal values.

Microgenetic development is interpreted by Billett as individuals' (teachers' and students') moment-by-moment construction of socially derived knowledge, derived through routine and non-routine problem solving. The nature of teaching as a profession is reflected in the relative autonomy assumed within the walls of the classroom, where teachers' decisions are constantly being made or revised on the basis of a continuous flow of new information. The instantaneous nature of many decisions is likely to be influenced to a greater or lesser extent by the teacher's internalised sets of values.

Ontogenetic development includes individuals' personal life histories, socially determined, which furnish the knowledge with which to interpret stimuli; this development includes participation in multiple overlapping communities.

This analysis points to the different sources of influence on teachers' values. Billett's categorised knowledge is a powerful indicator of how different knowledge at these five levels can impinge on and influence teachers' values in the classroom.

2.3 Bishop's (1988) socio-cultural dimension and its levels

My research context has been in the field of culture, and especially with considering mathematics as a form of cultural knowledge. When we are considering how to develop values in relation to mathematical thinking, I also believe we need to keep in mind the socio-cultural dimension of mathematics education. This dimension influences the values of mathematical thinking at five levels, which are similar but different to Billet's levels.

1. Cultural level – the overarching culture of the people, their language, their mathematics, their core values. In Billet's levels he combined together the cultural and the societal, which I believe in the case of mathematics education is not helpful. Evidence from research at the cultural level shows how different ethnomathematical ideas are not necessarily related to similar societal structures. Ethnomathematics points to cultural influences on mathematical thinking.
2. Societal level – the social institutions of the society, their goals, and their values regarding mathematics. In many societies mathematics education is a contested field with many proponents of different educational 'solutions' vying for publicity and academic advantage. They inevitably affect what is considered to be important mathematical thinking, and who is capable of doing it.
3. Institutional level – the educational institutions' values and the place of mathematics within them. At this level we can see the ways institutional values influence the curriculum, the timetable and even the allocation of space to each subject. These values also affect the development of mathematical thinking in different groups of students.

4. Pedagogical level – the teachers' values and decisions, the classroom culture of mathematical thinking. This is the same level as Billett's 'community of practice', and I have to confess that I prefer Billett's description of this level, as it emphasises the contribution of teacher and students to the classroom knowledge culture.
5. Individual level – individual learners' values and goals regarding mathematics, and mathematical thinking, which can differ markedly, and which do not necessarily follow the teachers' values and goals.

Thus I will draw on these three perspectives in the rest of this talk, and in particular I will assume that my ideas about values regarding mathematical thinking are:

1. Concerned with developing metacognition
2. Located within the socio-cultural dimension
3. Focused on the community of practice in the classroom.

3. Values and mathematical thinking

Now we turn to the values problem stated in Section 1 above. Building on the above analysis, I realised firstly that it was necessary to distinguish between three kinds of values:

- Mathematical values: values which have developed as the knowledge of mathematics has developed within any particular culture.
- General educational values: values associated with the norms of the particular society, and of the particular educational institution.
- Mathematics educational values: values embedded in the curriculum, textbooks, classroom practices, etc. as a result of the other sets of values.

My research approach to values and mathematical thinking has been to focus on mathematical values, and on the actions and choices concerning them (see Bishop, 1988, 1991, 1999). In my work I have used White's (1959) three component analysis of culture:

- Ideological component: composed of beliefs, dependent on symbols, philosophies,
- Sentimental (attitudinal) component: attitudes, feelings concerning people, behaviour,
- Sociological component: the customs, institutions, rules and patterns of interpersonal behaviour.

So how are these components interpretable in terms of mathematical thinking?

3.1 The Ideological component of Mathematical values

In regards to this component of the Mathematical culture, I argued (Bishop, 1988, 1991) that the critical values concern Rationalism and Objectism.
Valuing Rationalism means emphasising argument, reasoning, logical analysis, and explanations, arguably the most relevant value in mathematics education.

Ask yourself as a teacher:
Do you encourage your students to argue in your classes?

Do you have debates?
Do you emphasise mathematical proving?
Do you show the students examples of proofs from history (for example, different proofs of Pythagoras' theorem)?
Valuing Objectism means emphasising objectifying, concretising, symbolising, and applying the ideas of mathematics.

Ask yourself:
Do you encourage your students to invent their own symbols and terminology before showing them the 'official' ones?
Do you use geometric diagrams to illustrate algebraic relationships?
Do you show them different numerals used by different cultural groups in history?
Do you discuss the need for simplicity and conciseness in choosing symbols?

3.2 The Sentimental (Attitudinal) component of Mathematical values

In regards to this component, the important values are Control and Progress.

Valuing Control means emphasising the power of mathematical and scientific knowledge through mastery of rules, facts, procedures and established criteria.

Ask yourself:
Do you emphasise not just 'right' answers, but also the checking of answers, and the reasons for other answers not being 'right'?
Do you encourage the analysis and understanding of why routine calculations and algorithms 'work'?
Do you always show examples of how the mathematical ideas you are teaching are used in society

Valuing Progress means emphasising the ways that mathematical and scientific ideas grow and develop, through alternative theories, development of new methods and the questioning of existing ideas.

Ask yourself:
Do you emphasise alternative, and non-routine, solution strategies together with their reasons?
Do you encourage students to extend and generalise ideas from particular examples?
Do you stimulate them with stories of mathematical developments in history?

3.3 The Sociological component of Mathematical values

In regards to this component, the important values are Openness and Mystery.

Valuing Openness means emphasising the democratisation of knowledge, through demonstrations, proofs and individual explanations.
Ask yourself:

Do you encourage your students to defend and justify their answers publicly to the class?
Do you encourage the creation of posters so that the students can display their ideas?
Do you help them create student math newsletters, or web-pages, where they can present their ideas?

Valuing Mystery means emphasising the wonder, fascination, and mystique of mathematical ideas.
Ask yourself:
Do you tell them any stories about mathematical puzzles in the past, about for example the 'search' for negative numbers, or for zero?
Do you stimulate their mathematical imagination with pictures, artworks, images of infinity etc.?

These then are what I believe to be the crucial values underpinning the development of mathematical thinking in the classroom. I think we will make good progress in solving our problems if more research is devoted to investigating ways of developing these values in our teachers, so that they can develop them in their students.

4. Values, Mathematical Thinking and Lesson Study

Researching values development is no easy matter, but Lesson Study is an excellent method for studying the development of values in the classroom. In our Values and Mathematics Project (VAMP) we already used a version of lesson study, but without trying to affect the teachers' plans for their lessons.

1. The teachers told us before the lessons what values they thought they were going to develop.
2. We observed and recorded the lessons
3. We interviewed the teachers after the lessons to have them explain what they thought they had achieved.

More details of this research can be found at:
http://www.education.monash.edu.au/research/groups/smte/projects/vamp/vamppublications.html

For a full lesson study of mathematical thinking values, it would be necessary to plan together with the teachers what values they would try to develop.
The teaching ideas earlier would be very appropriate for this. It would be important for the experiment to go over a group of lessons, as values could hardly be developed in one lesson.

5. Conclusions for research

1. With any design and development research it is essential to have good theories to support and structure the work
2. Mathematical thinking has been studied in many ways, but in relation to values it is important to consider it as an aspect of meta-cognition.

3. The context for the research should be the classroom, as it is there that the community of practice significantly influences the meta-cognitive aspects of mathematical thinking.
4. Equally important to consider in this research is the socio-cultural context, as any educational values are embedded in the culture of the society.
5. Lesson study is an excellent research approach for studying any experimental educational development.
6. It is particularly appropriate for studying values development.
7. However there need to be a series of lessons studied as values do not develop in the space of one lesson.
8. Finally the teachers need special support in this research, as values teaching involve the teacher's pedagogical identity, which must be respected (Chin, Leu & Lin, 2001).

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TEACHERS' MATHEMATICAL THINKING

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Introduction

In my presentation at the Tokyo 2006 APEC symposium I demonstrated that mathematical thinking is important in three ways.

- Mathematical thinking is an important goal of schooling.
- Mathematical thinking is important as a way of learning mathematics.
- Mathematical thinking is important for teaching mathematics.

I spent most of that presentation discussing the first two dots points, and only discussed the third point with one example. In this presentation, I will discuss the third point in more depth. I ended my presentation at the last symposium with these comments:

"For those us who enjoy mathematical thinking, I believe it is productive to see teaching mathematics as another instance of solving problems with mathematics. This places the emphasis not on the static knowledge used in the lesson as above but on a process account of teaching. In order to use mathematics to solve a problem in any area of application, whether it is about money or physics or sport or engineering, mathematics must be used in combination with understanding from the area of application. In the case of teaching mathematics, the solver has to bring together expertise in both mathematics and in general pedagogy, and combine these two domains of knowledge together to solve the problem, whether it be to analyse subject matter, to create a plan for a good lesson, or on a minute-by-minute basis to respond to students in a mathematically productive way. If teachers are to encourage mathematical thinking in students, then they need to engage in mathematical thinking throughout the lesson themselves."

The first announcement for the December 2006 Tokyo APEC conference states that a teacher requires mathematical thinking for analysing subject matter (p. 4), planning lessons for a specified aim (p. 4) and anticipating students' responses (p. 5).These are indeed key places where mathematical thinking is required. However, in this section, I concentrate on the mathematical thinking that is needed on a minute by minute basis in the process of conducting a good mathematics lesson. Mathematical thinking is not just in planning lessons and curricula; it makes a difference to every minute of the lesson. In this analysis, I aim to illustrate how strong and quick mathematical thinking provides the teacher with many possible courses of action. The course of the lesson, though, is then determined by how the teacher weighs up the possibilities which he or she sees. The mathematical possibilities are considered along with knowledge of students' mathematical understandings and needs and with pragmatic factors (eg those associated with keeping the lesson on track), and a choice is made. These decisions determine the course of a lesson.

We now examine the mathematics used by two teachers when their classes tackle the 'spinners game'. After this, I also report on experiences when the problem was adapted and used in a primary teacher education class.

Irene's lesson on the Spinners Game

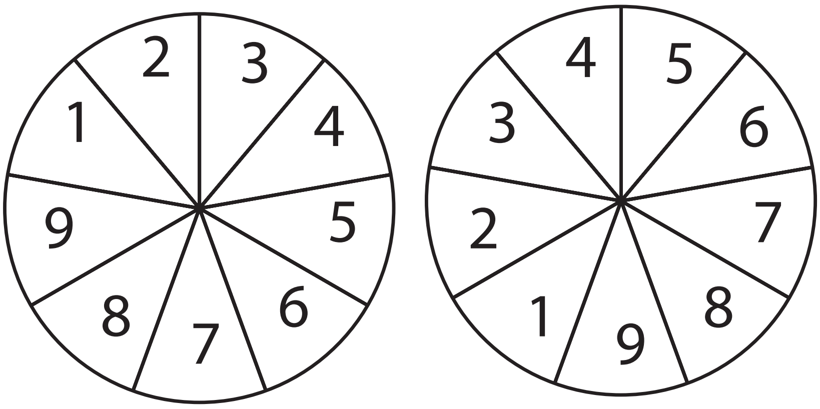

Figure 1. Equipment for the spinners game

The spinners game was first discussed in Chick and Baker (2005) and is based on their classroom observations and interviews with the teachers. This account of two classroom uses is reproduced with adaption from Chick (2007) with permission, and additional points relevant to this presentation have been inserted. Irene, an experienced teacher, and Greg, who was in only his second year of teaching, were Grade 5 teachers in the same school. They had chosen to use a spinner game suggested in a teacher resource book (Feely, 2003). The spinner game used two spinners divided into nine equal sectors, labelled with the numbers 1-9 (see Figure 1). The worksheet instructed students to spin both spinners, and add the resulting two numbers together. If the sum was odd, player 1 won a point, whereas player 2 won a point if the sum was even. The first player to 10 points was deemed the winner. Students were further instructed to play the game a few times to "see what happens", and then decide if the game is fair, who has a better chance of winning, and why (Feely, 2003, p. 173). The teacher instructions (Feely, 2003, p. 116) included a brief suggestion about focusing on how many combinations of numbers add to make even and odd numbers but did not provide any additional direction.

This game can offer worthwhile learning opportunities associated with sample space, fairness, long-term probability, likelihood, and reasoning about sums of odd and even numbers. The significant issue here, especially in the absence of explicit guidance from the resource book is *how* these learning opportunities can be brought out. Although it is not written in the teachers' resource book, the spinners game has an interesting twist. Analysis of the sample space shows that the chances of Player 2 (even) winning a point is 41/81 compared to 40/81 for Player 1 (odd). Player 2 is therefore theoretically more likely to win, however this miniscule difference in likelihood implies that the game's theoretical unfairness will not be evident when playing "first to ten points". We cannot tell whether the authors of the resource book chose this narrow difference deliberately or accidentally. Our interest here is in the teachers' mathematical thinking as they implemented the activity in the classroom.

Irene started the spinners' game late in a lesson. Most students had played the game for a few minutes before Irene began a short class discussion. She asked the class if they thought it was a fair game. Discussion ensued, as students posed various ideas without any of them being completely resolved. For instance, someone noted that fairness requires that players play by the rules of the game. Most of the arguments about fairness were associated with the number of odds and evens, both in terms of the individual numbers on the spinners (there are 5 odds and only 4 evens on each spinner) and in terms of the sums. One student neatly articulated an erroneous parity argument, that since "odd + odd = even and even + even = even but odd + even = odd, therefore Player 2 has two out of three chances to win". Irene said she was not convinced about the "two out of three", but she agreed the game was unfair.

The student's presentation of this argument, which Irene suspects is not valid, requires her to make a decision as to whether it should be pursued, or passed over quickly in favour of something else. She might, for example, have presented (or sought from a student) another erroneous argument along the same lines but which takes into account the fact that odd + even and even + odd occur in different ways: "odd + odd = even and even + even = even but odd + even = odd *and even + odd = odd*, therefore *both* players have two out of *four* chances to win". Presenting this argument would have emphasised that the different orders are important, but the new argument has the same failing as the first argument. It does not take into account that there are different numbers of odd and even numbers. Instead Irene might have decided to highlight just this failing of the student's argument, showing, for example that odd + odd is more likely than even + even. The several possibilities for responding to the argument as well as the possibility of simply passing over it quickly, as she chose to do, must be identified and evaluated in just a few seconds as the classroom discussion proceeds.

Good decisions would seem to be enhanced when teachers see the mathematical possibilities quickly and evaluate them from a mathematical point of view (what important mathematical principles/processes/strategies/attitudes would the students learn from this). However, decision making also needs to be informed by knowledge of how the students will respond, and by attention to practical aspects of the lesson, including the time available. For Irene, the necessity to finish the spinners game in the few remaining minutes of the lesson might have been the over-riding consideration.

Irene then allowed one of the students to present his argument. At the start of the whole class discussion this student had indicated that he had not played the game at all but had "mathsed it" instead, and at that time Irene made a deliberate decision to delay the details of his contribution until the other students had had their say. He proceeded to explain that he had counted up all the possibilities, to get 38 even totals and 35 odd totals. Although this was actually incorrect, Irene seemed to believe that he was right and continued by pointing out that this meant that “it’s [the game is] not *terribly* weighted but it *is slightly* weighted to the evens”. Irene then asked the class if their results bore this out, and highlighted that although the game was biased toward Player 2 this did not mean that Player 2 would always win.

As suggested earlier, the spinner game provides the opportunity to examine sample space, likelihood, and fairness. Given the impact of time constraints on Irene’s lesson, sample space was not covered well, although she believed that the student who had “mathsed it” had considered all the possibilities. This highlights a contrast between her knowledge of his capabilities and of the details of the content with which he was engaged. On the other hand, her content knowledge was sufficient for her to recognise the significance of the small difference between the number of odd and even outcomes and its impact on fairness. Irene led a good discussion of the meaning of fairness and the magnitude of the bias, and its consequence for the ‘first to ten’ aspect. Given the short time available to end the lesson, it may have been a wise decision to ignore the errors in the student’s sample space and go on to what Irene probably saw as the main point: that the difference in likelihood is very small, and that even if there is a bias students would not have been able to reliably detect it in the ‘first to ten’ game.

In considering Irene’s lesson, we see that its path is determined by many small decision points: who to call on next, whether to check the student’s list of outcomes or simply believe him because he is a good student, whether to pursue the errors in the parity argument etc. These decisions are influenced by factors relating to the mathematics (as perceived during the flow of the lesson by the teacher), factors relating to the students’ current knowledge and factors relating to the pragmatic conduct of the lesson (e.g. how much time is left). This is illustrated in Figure 2.

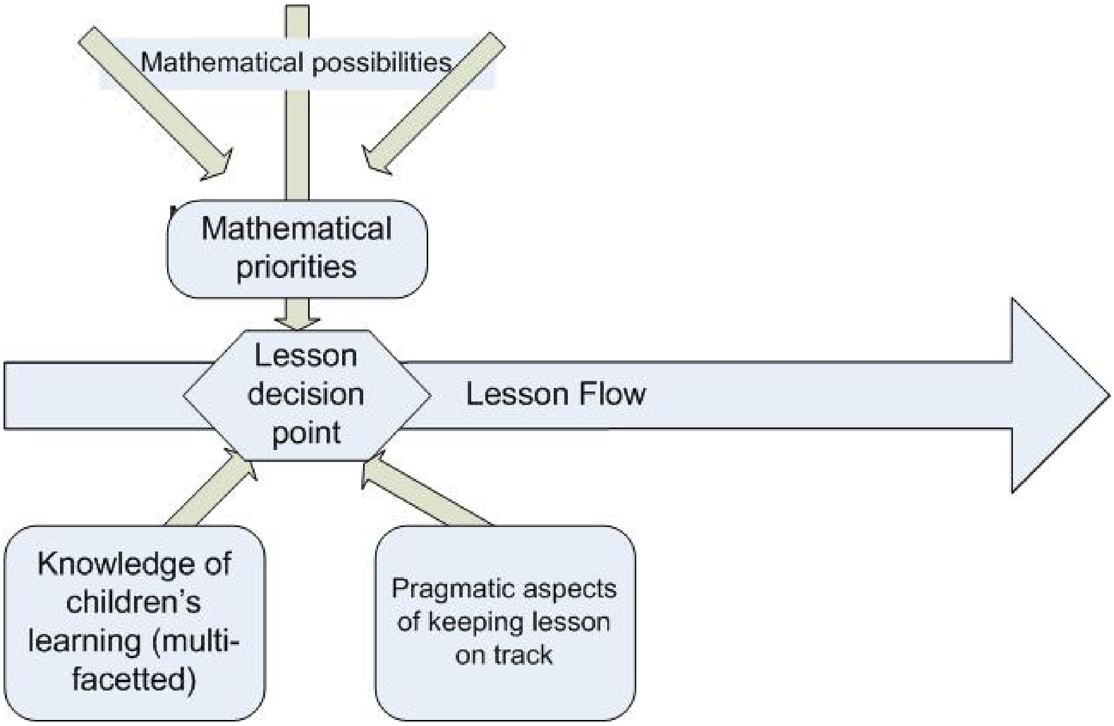

Figure 2. *Decision points in lessons are influenced by many factors.*

Greg's lessons on the Spinners Game

In Greg's class, students played the spinners game at the end of a lesson, and students put forward various ideas about whether the game was fair. During this time, Greg decided that the next lesson should be spent on finding the sample space. Greg then devoted nearly half of his second lesson to an exploration of the sample space. As reported in Chick and Baker (2005) he tightly guided the students in recording all the outcomes and could not deal with alternative approaches. He asked the students to calculate the probabilities of particular outcomes, which was helpful in highlighting the value of enumerating the sample space, but detracted from the problem of ascertaining whether even or odd outcomes were more likely. Students eventually obtained the "40 odds and 41 evens" conclusion, at which point Greg stated that because the "evens" outcome was more likely the game was unfair. There was, however, no discussion of the narrowness of the margin, or the difficulty of confirming this result empirically through the 'first to ten' aspect of the game.

In summary, Greg was much more thorough than Irene in his consideration of sample space, but also very directive. Neither teacher seemed aware of all that the game afforded in advance of using it, as evidenced by the way it was used, although Greg recognised the scope for examining sample space part way through the first lesson. Both teachers were, however, able to bring out some of the concepts in their use of the game, with Irene having a good discussion of the meaning of fairness and the magnitude of the bias, and Greg illustrating sample space and the probability of certain outcomes. An important observation needs to be made here. The teacher guide that was the source of the activity gave too little guidance about what the spinner game afforded and how to bring it out. Even if such guidance had been provided, there is also still the miniscule bias problem inherent in the game's structure that affects what the activity can afford. It is very difficult to convincingly make some of the points about sample space, likelihood, and fairness with the example as it stands. It can be done, but the activity probably needs to be supplemented with other examples that make some of the concepts more obvious (see, e.g., Baker & Chick, 2007). This highlights the crucial question of how can teachers be helped to recognise what an example affords and then adapt it, if necessary, so that it *better* illustrates the concepts that it is intended to convey.

Interestingly, in both classes the students did not—indeed could not in any reasonable time frame —play the game long enough for the slight unfairness to be genuinely evident in practice, yet most students claimed that the game was biased towards even. This may have occurred because the incorrect parity argument made them more aware of the even outcomes than the odd ones.

The observers were surprised by the tight way in which Greg controlled the method by which the outcomes were enumerated. He wanted to see the 81 outcomes, along the lines of the enumeration on the left hand side of Figure 3, although in an array setting out. Greg seemed constrained by his mathematical knowledge, having only one way to think of the sample space—via exhaustive enumeration. When a student offered an erroneous suggestion which could have been readily adapted to a more elegant and insightful method, he did not encourage or discuss it. In fact, there are many bridges between the totally routine method of writing out 81 outcomes and counting how many totals are even or odd, and insightful ways which give the answer quickly. At the top right hand side of Figure 3, for example, is one of the bridges. As they begin work on the exhaustive enumeration on the left hand side, students might be encouraged to note the patterns – alternating evens and odds for a fixed first choice, the EOEOEOEOE pattern when the first choice is odd and the OEOEOEOEO pattern when the first choice is even. These patterns are easily explained by students, and they can be readily utilised to find the how many even and odd sums there are, either by addition or by multiplication as outlined in the figure. The tree diagram approach at the bottom of Figure 3 would be too sophisticated for Greg's young students, since it relies on more strongly combinatorial thinking, but a version of it might be reached after experience with the patterns above.

Enumeration of 81 separate outcomes			
First	**Second**	**Sum**	**Sum**
1	1	2	E
1	2	3	O
1	3	4	E
1	4	5	O
1	5	6	E
1	6	7	O
1	7	8	E
1	8	9	O
1	9	10	E
2	1	3	O
2	2	4	E
2	3	5	O
2	4	6	E
2	5	7	O
2	6	8	E
2	7	9	O
2	8	10	E
2	9	11	O
3	1	4	E
3	2	5	O
3	3	6	E
3	4	7	O
3	5	8	E
etc	etc	…	…

Enumeration mixing number and parity considerations					
First	**Second**	**Sum**	**First**	**Second**	**Sum**
Odd (eg 1)	1	E	Even (eg 2)	1	O
	2	O		2	E
	3	E		3	O
	4	O		4	E
	5	E		5	O
	6	O		6	E
	7	E		7	O
	8	O		8	E
	9	E		9	O
5 evens and 4 odds			4 evens and 5 odds		
The 5 odd numbers contribute 25 evens and 20 odds, and the 4 even numbers contribute 16 evens and 20 odds, making 41 evens and 40 odds altogether.					

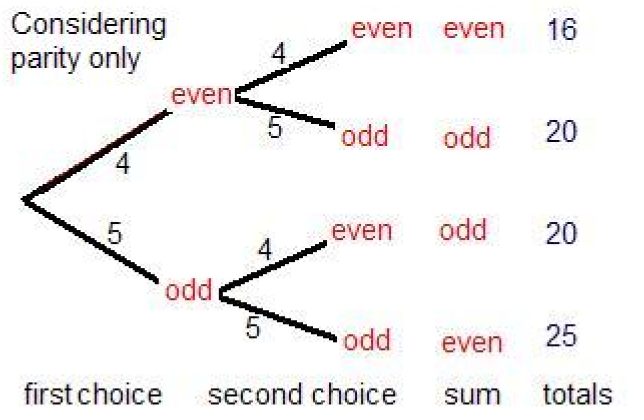

Figure 3. *Three different ways of counting numbers of odd and even spinner totals*

In considering why Greg made his decision to focus his lesson on finding the 81 element sample space in one particular way, it is again likely that his decision is influenced by judgements about mathematical factors, factors related to the students and their current knowledge and pragmatic factors related to the lesson. Greg decided in the first lesson that he would allocate the second lesson to finding the sample space, so it was a priority for him, and he taught it thoroughly. Whereas Irene's treatment of sample space appeared rushed in response to a shortage of time, Greg decided that this was sufficiently important for a second lesson. His priority given to the idea of sample space is also evident in the observation that he did not focus only on the spinner game, but used the sample space to find the probability of events unrelated to the initial spinner game.

As is illustrated in Figure 2, mathematical priorities can only be chosen from the mathematical possibilities that are perceived by the teacher. Consequently, it may be that Greg's focus on one way of finding the sample space was because he was not aware of other ways, or was uncertain of their validity. On the other hand, it may have been a more active prioritising. He may have seen value in teaching students about systematic listing, and wanted students to go through that process very thoroughly, getting a real 'feel' for how to go through the cases one by one. From yet another point of view, Greg may have judged that the full, very routine, case-by-case enumeration was at an appropriate level for his target group in that class, and so he may have selected the method as optimal for the whole class, even if not for each individuals.

This is all speculation, even though Greg was interviewed about his lesson (which contained many other features). It is simply not possible for teachers to thoroughly explain each of the myriad decisions that are made in the course of any one lesson. The point of this discussion, though, is that at any stage in the lesson, Greg was aware of certain mathematical possibilities. These may have resulted from deep or superficial insight into the spinners game; they may be numerous or sparse; they may be mainly procedural or extend to strategic thinking etc. To make a decision on how to respond to a student's question or a mathematical problem arising in the conduct of the class, Greg has to set priorities and act on them. In this way, we see that a teacher's mathematical knowledge (conceptual, procedural, strategic etc) sets the choices and so is very important, but good decision making also depends on teachers being able to make good choices amongst them, in the light of progressing the main aims of the lesson.

Helen's lesson on the spinners game

Even when lessons are videotaped and teachers are interviewed after the lesson, much of the mathematical thinking upon which teachers make decisions about the paths of lessons remains hidden. For this reason, the next example is about a discussion on an episode in a teacher education class, which we discussed together on several occasions.

Helen teaches pre-service primary teacher education students and is a highly accomplished mathematical thinker. She had observed the lessons of Irene and Gary, and decided to use the spinners game in class. She wanted student teachers to analyse what mathematical learning it could generate and how. To simplify and also to extend the game, Helen changed the numbers on the two spinners (see Baker and Chick (2007) for examples).

On one occasion, Helen's class used two spinners labelled with 0, 1, 2 and 3. This small change, selected by Helen to simplify the game, caused a new complication. Many of her students began the enumeration, but halted when they needed to decide whether the sum of 0, obtained by throwing 0 on each spinner, was an even number or an odd number, both or neither. This turned what was intended to be a short mathematical episode using a simplified spinners game used, to an unpredicted query about odd and even numbers.

At this point, Helen faces a decision. Once again it will be informed by her knowledge of mathematics and her mathematical thinking during the lesson, and by weighing the priorities for the lesson. This will be discussed below. However, it is worth observing first that Helen had not predicted the evenness of 0 would be such an obstacle to the progress of this lesson. In future use of the spinners game, having this additional knowledge of students (urther pedagogical content knowledge), she may avoid using the number 0 on the spinners so that the lesson proceeds without this obstacle, or she may deliberately choose it to uncover these misconceptions.

Addressing the apparently simple question of whether 0 is even or odd or neither or both, draws again on mathematical knowledge and pedagogical content knowledge (Shulman, 1986, 1987) working in tandem. The student teachers were very familiar with the fact that 2, 4, 6, 8, 10, 12, etc are even numbers. Why would they query whether 0 is even, and what would convince them that it is? Possibly the reason for the difficulty is that students draw on intuitive meanings for 'even', rather than a mathematical definition. For example, they may associate an even number with the possibility of pairing up. If there is an even number of children in our class, we can go for a walk arranged in pairs. If there is an odd number of children, there will be one left over, as illustrated in Figure 4.

Figure 4. *An even number of children can walk in pairs, but not an odd number.*

This informal interpretation of 'even number' is difficult to apply to decide whether 0 is even or odd, because whilst there is certainly not 'one left over', there are no pairs either. Kaplan (1999) discusses difficulties like this. Alternatively, students who draw just on the list of examples to decide if a number is even or odd (2, 4, 6, … etc) have no way of knowing whether 0 should really be on the list or not, when there is no principle to guide them. They know 0 is special – is this another way in which it is special?

Helen was keen to draw her students' attention to the mathematical definition of an even number, but she reported that she immediately saw two possibilities. She could say that an even number is defined to be an integer which is exactly divisible by 2 or that an even number is defined to be an integer that is equal to 2 times an integer. This might seem a small difference, but Helen chose the second version because of her previous experience with the awkwardness in teaching associated with discussions of dividing *by* zero. Even though the test for evenness does not involve dividing *by* zero (but dividing *by* 2), Helen avoided the division explanation because she felt students may confuse the situations. In other words, she presented students with finding whether there is an integer satisfying the first rather than the (equivalent) second equation below:

$$0 = 2 \times ?$$

$$0 \div 2 = ?$$

[In neither case could she avoid the likely obstacle of students' uncertainly about whether 0 is an integer.] Here we see that Helen's strong mathematical knowledge and her ability to see the mathematical possibilities quickly presented her with possibilities. Her pedagogical content knowledge (in this case of likely students' difficulties) guided her choice.

Was it best to pause to discuss why 0 is even? Helen could have just asserted that 0 is even and moved the lesson back on the track of investigating the fairness of the spinners game. When reflecting on this question, Helen asserted that the diversion was useful because it enabled her to clarify some fundamental misunderstandings about zero and to show how mathematical concepts are determined by definitions. Here, we see that Helen justifies her choice in terms of her understanding of important principles of doing mathematics – in this case the role of definitions in mathematics. More fundamentally, it seems to reveal a predisposition on Helen's part to avoid having students see mathematics as arbitrary and without reason.

After her observations of the lessons of Irene and Greg, Helen and her colleague published a suggested teaching sequence for primary classes using the spinners game (Baker and Chick, 2007). The spinners she suggests have no zeros. Her suggested sequence begins with a pair of spinners each with just 3 digits, arranged so that there is a strong enough bias to be evident in empirical trials. Students begin by finding this empirical experience of the bias, tallying class results. Students then draw up the sample space and compare theoretical probabilities to empirical class results. They discuss variations between theory and experiment. The pair of spinners chosen are biased towards odd totals (they do not have the same numbers on each spinner – see mathematical note below. Helen has selected these spinners so that the false parity arguments give an obviously wrong answer. This is a very substantial example of mathematical thinking being used in lesson planning, again in concert with pedagogical content knowledge – in this case knowledge of students' false arguments. Helen's suggested lesson sequence then moves back to the original spinners problem. this gives experience in finding a large sample space systematically and subtleties of comparing theoretical and empirical results when the bias is small. Finally students create their own spinners and discuss what they designed the spinners for, how unfair the game is, and what is likely to happen if they play the game many times.

Conclusion

At the beginning of this paper, I drew an analogy between teaching a mathematics lessons and solving a real world problem with mathematics. I noted that in order to use mathematics to solve a problem in an area of application, mathematics must be used in combination with knowledge from the area of application. In the case of teaching mathematics, the area of application is the classroom and so the teacher as 'mathematical problem solver' has to draw on general pedagogy as well as mathematical pedagogical content knowledge to contribute to the solution. As will many problems in areas of application of mathematics, these teaching problems need to be solved in an environment that is rich in constraints: short lesson times, inadequate resources at hand, etc. In the teachers' role of analysing subject matter, designing curricula or in creating a plan for a good lesson, solving the problem can occur with adequate time for reflection, testing ideas and reconsidering choices. However, in the course of a lesson, this mathematico-pedagogical thinking happens on a minute-by-minute basis, with the aim of responding to students in a mathematically productive way. If teachers are to encourage mathematical thinking in students, then they need to engage in mathematical thinking throughout the lesson themselves, but this mathematical thinking is under severe time pressure.

In the conduct of a lesson, teachers see various mathematical possibilities. Some teachers will see more than others in any given situation and some of the possibilities that teachers see may not be correct. The process of choosing amongst these possibilities, which again occurs on a minute by minute basis, will be guided by the deep knowledge of the students (the actual current mathematical knowledge of these students as well as thinking typical of students like these), operating under the constraints of teaching a lesson in a fixed time to achieve an identified goal. Teachers who are stronger mathematical thinkers will see more possibilities, and in the moment when a decision needs to be made, their choices will be better informed by teaching underlying mathematical processes and strategies.

A mathematical note

Solving the problem of bias in the spinners game is a nice example in algebraic factorisation, with surprising results.

If there are n even numbers and m odd numbers on the spinners, then there are $n^2 + m^2$ ways of getting an even total, and $2mn$ ways of getting an odd total (see Figure 5). Since $n^2 + m^2 - 2mn = (n-m)^2 \geq 0$ we can conclude that

(i) if $n = m$ then the spinner game is fair

(ii) otherwise, there is always slightly more chance of getting an even number.

Moreover, if the numbers on the spinners are consecutive whole numbers, then *n* and *m* will either be equal or differ by 1 (ie n –m = 0 or |n-m| = 1). This means that the number of even sums will always be equal to, or one more than the number of odd sums. In this way, we see that the very close comparative probabilities of the original spinners game (41/81 and 40/81) are typical of having consecutive numbers on the spinners.

To generalise further, if there are n_1 evens and m_1 odds on the first spinner and n_2 and m_2 on the second spinner (respectively) then there are $n_1n_2 + m_1m_2$ even sums and $n_1m_2 + n_2m_1$ odd sums. Are evens or odds more likely to be thrown? Calculate the difference in number of outcomes:

$$n_1m_1 + n_2m_2 - (n_1m_2 + n_2m_1) = (n_1 - m_1)(n_2 - m_2)$$

This means that if evens are more prevalent on both spinners OR odds are more prevalent on both spinners (ie the two factors in the final product have the same sign), then the game is biased in favour of the even sums. Alternatively, it evens are more prevalent on one spinner and odds more prevalent on the other spinner (ie the two factors in the product have opposite signs), then the game is biased in favour of odds.

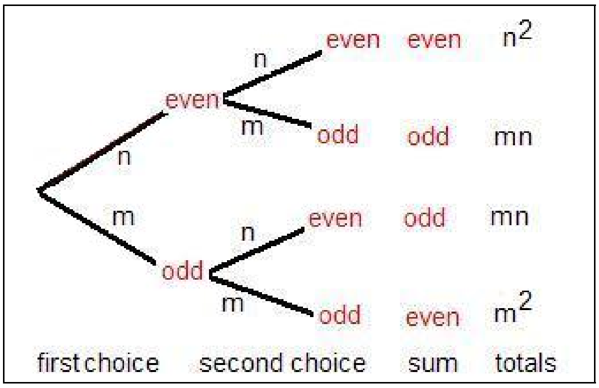

Figure 5. *Odd and even spinner totals from spinners with n even and m odd numbers*

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MATHEMATICAL THINKING IN JAPANESE CLASSROOMS

Abraham Arcavi

Introduction: personal background and intention

As a young student, I have experienced mathematics classrooms at the elementary and secondary school level and at the university level in Argentina. After that, I have experienced mathematics classrooms as a graduate student in Israel. In both countries, I taught mathematics at junior high school and high schools. In Israel and the USA, I was engaged in curriculum development, teacher education and research on teaching and learning. It is against that variegated, yet all-Western, background that I was exposed to the fascinating microcosms of Japanese mathematics classrooms at the elementary school level. Before I went to CRICED, at Tsukuba University to work with Professor Masami Isoda for four months, I had read research describing lesson study in Japan (e.g. Fernandez and Yoshida, 2004, and other sources) which focused on comparing and contrasting Japanese (and other Asian classrooms) with American classrooms (e.g. Stevenson and Stigler, 1992, Stigler and Hiebert, 1999). I had also watched and analyzed an algebra and a geometry lesson (published by TIMSS in 1999), and on their basis, an in-service workshop for teachers and teachers of teachers was designed and implemented (Arcavi & Schoenfeld, 2006) drawing on ideas from the Teacher Model developed by Schoenfeld (1998). However, I think I was able to fully appreciate the teaching in mathematics classrooms in Japan only through the non-mediated experience (except for the simultaneous translation) of "being there", watching how lessons evolve, following children's work and discussions, talking to teachers and researchers and sensing the common pedagogical and mathematical characteristics of all the lessons I saw.

If we take the statement that "there is nothing more practical than a good theory" (Lewin, 1952, p. 169) and attempt to formulate its symmetrical version, we may propose that "there is nothing more theoretical than a good practice". This may make little sense as stated, however, it may suggest that a exemplary practice can be a powerful source for theorizing, which in turn may help understand the practice, especially a teaching practice. Our field has many learning theories, however there are not so many instructional theories. It is with the intention of contributing to instructional theories, that I would like to briefly share what I have learned from mathematics classrooms in Japan.

Mathematical Thinking – A Japanese view

"Mathematical thinking is the "scholastic ability" we must work hardest to cultivate in arithmetic and mathematics courses… [it] is even more important than knowledge and skill, because it enables to drive the necessary knowledge and skill" (Katagiri, 2006, p.5). Moreover, "mathematical thinking allows for (1) an understanding of the necessity of

using knowledge and skills (2) learning how to learn by oneself, and the attainment of the abilities required for independent learning" (Katagiri, 2006, p.6)
According to this philosophy, advancing mathematical thinking includes the development of:

- '**attitudes**'– intellectual predispositions towards doing mathematics and solving problems, including perspectives on what mathematics and mathematical activity are,
- '**contents**'– concepts, properties, interrelationships, and
- '**methods**'– inductive and deductive reasoning, analogical thinking, generalization, specialization, symbolization.

This rich view of what constitutes mathematical thinking drives teaching and was clearly reflected in all the lessons I saw and analyzed.

In the following, I concentrate on a particular aspect of the teaching practice: "teacher" actions, decision making, lesson crafting and the classroom setting which are aimed at the development, support and encouragement of sound and independent mathematics thinking. In other words, I describe what Japanese teachers actually do, how they do it, in order to engage students in thinking and learning while they are 'doing' mathematics, and how do teachers connect to students "en masse" in a fruitful and unconstrained way? (Lampert, 2001, p. 424).

Characteristics of the lessons

The following are some of the common characteristics of mathematical lessons in elementary school, which I found to be at the core of supporting mathematical thinking, and which are the result of purposeful planning and crafting by the teachers.

Coherence

All the lessons have a "story" – "A good story is highly organized; it has a beginning, a middle, and an end, and it follows a protagonist who meets challenges and resolves problems that arise along the way. Above all, a good story engages the reader's interest in a series of interconnected events, each of which is best understood in the context of the events that precede and follow it" (Stevenson & Stigler, 1992, p. 177). In other words, each lesson evolves and revolves around a central mathematical problem on which students work bringing in their common sense, their knowledge from outside mathematics, their findings from investigating the problem and their ongoing building on the ideas they produce in situ. The teacher leads students to apply knowledge and ideas that emerge during the lesson. The work has a unifying mathematical thread and sometimes the teacher creates an atmosphere of "suspense" around features of the problem, which fuels interest and maintains students engaged and active.
Coherent lessons that pursue central ideas around a meaningful and interesting problem are related to all the three components of mathematical thinking described above: attitudes, contents and methods. Whole lessons which pursue a central problem have the potential to nurture a view of mathematics as a discipline that tackles complex and relevant problems, which take some time to solve and which include attempts that fail and attempts that succeed, alternative approaches, discussion and exchange of ideas. The content involved

in solving a central problem goes beyond the presentation of a skill or a concept, the teacher involves the students in both conceptual understanding and in procedural activities, which are interwoven and are at the service of each other. In the process students propose methods of work and apply different ways of reasoning.

The following is an example of a lesson centered around one problem “the unfolding of the cylinder” (Arcavi, 2001). This is the second of three lessons (allotted by the curriculum) on the unfolding of solids. During this class, the problem is to design models of unfolded cylinders and then to assemble them in order to check that they indeed yield a cylinder. The goals are that students learn interactively (with concrete materials and with other students) about the structural components of the intervening two-dimensional figures, their relative positions, and, in certain cases, the importance of careful planning and measurements. In the process, students exercise their imagination, spatial visualization abilities, and creativity. The lesson opens with the teacher reminding the class of a previous lesson on the unfolding of a tetrahedron, and asks to think about the shape of an unfolded cylinder. After the class worked on the problem for a while, the teacher invites students to share their drawings on the blackboard. The first proposal is the classical: a rectangle and two tangent circles attached to its largest sides (prototypically, the largest sides are the horizontal). The teacher takes the opportunity to analyze the figure with the class, and to make sure students understand and agree on all the details. Thus, by asking several questions, simple but very important issues are raised and discussed, like:

- the two circles (the bases of the cylinder) should be of the same size,
- the two circles should be tangent to the a corresponding pair of parallel sides of the rectangle (and not secant to them),
- the length of the tangent sides should be equal to the circumference of the circles (students recall the number π and the formula for calculating the circumference),
- the length of the other two sides of the rectangle (to which the circles are not attached) are unconstrained (short sides and long sides will yield short or slender cylinders respectively),
- the points of tangency could be anywhere on each of the opposite sides of the rectangle.

Once these issues were discussed, the teacher encourages the class to produce alternative plane models for an unfolded cylinder. The class begins to propose other models including slicing and re-attaching parts of the bases, and many other creative designs, many of which will not fold into a cylinder. At a certain point, the teacher encourages the students to actually cut their designs, attempt to fold them into a cylinder and see if they succeed. In case of failure, students are encouraged to analyse the sources of their erroneous designs. By the end of the class, all the models are displayed.

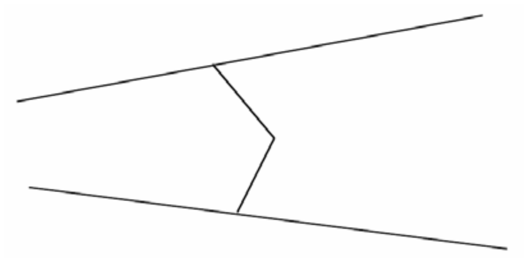
The “story” of this lesson has distinct parts: the discussion of the prototypical unfolding, the planning and design of alternatives, the practical work of assembling of the cylinders out of the models proposed, and the noticing and discussions of the failures. Coherence is not only a characteristic kept within the lesson (including the integration of visual reasoning with calculations, posing conjectures and checking them, analysing failed attempts and discussing each other’s solutions), but it is also related to the two other lessons this class has on the unfolding of solids.

Challenging problems

In most of the lessons I saw, there were instances in which I found myself solving mathematical aspects of the problem, as if I were a very engaged student participating in the class. I took this as an indication that the problems and discussions in these lessons are indeed mathematically interesting, challenging, and deep.

Consider, for example, a third grade lesson entitled “New ways of calculation” (for a detailed description see Arcavi 2007), in which the students are asked to calculate a series of multiplications of two numbers between 20 and 30 in which their unit digits add up to 10 (e.g. 23x27, 24x26, 25x25). As the lesson slowly unfolds, the teacher asked students to notice, record and communicate patterns (the way the exercises are handed in does not make the task of finding patterns a straightforward one), to propose an easy algorithm to perform these calculations, and to attempt to explain when and why it works. The new rule students discover and propose is that the result is 600 plus the result of the multiplication of the two digits, e.g. 23x27=600+3x7=621. Obviously, third grade students lack the algebraic tools to generalize and explain why the rule works, thus students work at the edge of their knowledge (or perhaps a little beyond that).

The following example is the central problem of the geometry class of the TIMSS video. “Replace the non-straight boundary dividing two pieces of land in the figure below



by a straight boundary line, while preserving the areas of the original pieces.”

I have shared this problem with many knowledgeable mathematics teachers and they worked for a while before finding a solution, agreeing that this is a very difficult problem to be given to 8th graders. Certainly, this challenging problem was given within a coherent sequence of lessons and was supposed to be a non-trivial application of a property studied in a previous lesson: that all the triangles with the same base, and whose third vertex lies on a line parallel to that base, have the same area (the “constant area property”).

The above problems are very different in nature, but they share some characteristics: they are not straightforward exercises, they require students to work at the edge of their knowledge, to explore, to discuss different approaches and to slowly device ways to make progress on the basis of mathematical content and different ways of reasoning.

The choice of problems like these implies that Japanese teachers feel very comfortable with their mathematical knowledge. But most importantly, by using these problems for an entire class, teachers enact their confidence that their students can and should engage with mathematical challenges and that they will be able to make progress.

Posing questions

The questions teachers pose to their students during the lessons and their questioning techniques are in consonance with the type of problems that teachers choose to be at the core of each lesson and at the service of solving them and learning from them. Many times the questions request explanations, arguments or counter-arguments. A salient characteristic of the lesson is to make sure that these explanations are fully understood by everybody, and one would expect the following line of questioning to attain this goal: requesting the student to repeat her explanation for the rest of the class, asking the class whether they understood it, asking who agrees or disagrees, and requesting other explanations. These are indeed part of the teacher's repertoire of questions. However, a technique which I found of great interest and importance is none of the above: after a student produces an explanation, an argument or proposes a conjecture the teacher asks the whole class who can explain such explanation or who can tell 'what is the thinking behind such explanation or proposal'. Such request nudges students to carefully learn to listen to each other, and before they can agree or disagree, to take the other's ideas and be able to replay and enact them as if they were theirs. Learning to listen to each other can be highly beneficial in developing the kinds of mathematical thinking the Japanese teachers are after in at least two different directions. Firstly, it may support the nurturing of empathic and caring relationships by conveying the message 'I take your idea and delve deep into it and its merits and sources'. Such a message is the precondition for the development of "academic civility" (Lampert, 2001, p. 431) in a classroom, within which all ideas are respected and valued and inspected mathematically. Secondly, by fully taking the other's perspective, one may be exposed to new ideas and forced to analyze them from within – on the one hand, this helps towards 'decentering' oneself and on the other hand, this may lead to re-inspect one's own knowledge, against the background of what was heard from others (Arcavi & Isoda, 2007). Thus, such a simple request from the teacher may be instrumental in supporting mathematical thinking.

There is another aspect to the questions teachers ask: the redefinition of the role of authority. In my observations, the teachers' authority is reflected in the decision about which task to focus on, which questions to pursue and when, how to distribute the right to speak and how to sequence the activities. The authority is not exerted in determining what is mathematically right or wrong, in this case the teacher deflects, as far as possible, such authority to the mathematics itself (Arcavi et al., 1998), placing a central role on the production of explanations and arguments to settle opposing results. This implies that erroneous answers are not immediately judged as incorrect, they have legitimate status until they are discussed against others. Building on students capacities to evaluate mathematical arguments and ideas places on them responsibility on their own learning and indeed supports the development of mathematical thinking.

Anticipation

Asking good open questions (such as requesting explanations, conjectures and proposals for strategies and ideas) constitutes, in more than one sense, a challenge for the teachers, because when students respond bringing in their proposals and ideas, teachers must do several things at once. Firstly, they have to perform an on site and very quick evaluation of the mathematical merit of the students' proposals, and this implies a solid mathematical background on the part of the teacher and a confidence to put it to use 'online'. Secondly, there must be an evaluation of the pedagogical possibilities that a student proposal affords and a decision regarding how to take advantage of it - this may imply changing the direction of the planned lesson and sometimes even some relinquishing of the control on the new directions the lesson may take. In my view, this is one the most difficult predicaments of the teaching profession. How do Japanese teachers cope with such situations? As far as I understand it, this issue is at the core lesson study: to study a lesson in depth and to implement it several times such that most students' reactions and proposals can be anticipated and only very few are new and surprising. Anticipated student reactions unload from the teacher the burden of on-site decision making. Furthermore, very fruitful student reactions which can contribute to the course of the lesson and which the teacher knows the students can produce them, can be stimulated and looked for at appropriate moments of the lesson.

Diversity

Much has been said about the ethnic (maybe also socio-economical) and cultural homogeneity of the Japanese society. Thus classroom realities are very different from those of the countries I know. Multiculturalism, multilingualism and social deprivation, which are pervasively reflected in classrooms in many Western countries are almost not known in Japan. However, there is another kind of diversity which is as present in Japanese classrooms as in their Western counterparts and which is no less of a challenge to teaching: children differ in their academic achievements and in their mastery of the subject. Japanese teachers cope with such diversity using different pedagogical approaches to a same topic, they have a proper pace and they harness all students' responses from the less to the more sophisticated (in that order) in the development of a lesson. However, no matter how competitive this society may be regarded by many, elementary schools do not track students and the teachers attend to all children in an impressively inclusive way. It is true that the respectful way in which Japanese people treat and address each other in ingrained in the culture and is the common and natural behavior: in contrast to what I have seen in many classrooms in other countries, I have not seen any Japanese teacher raising their voices or reprimanding students. Even when initial commotion was caused by the presence of visitors or by the mere playfulness of the children, the unrest quiets down by itself, mostly without the need of any intervention. Thus, the atmosphere teachers manage to induce in their classrooms is very propitious for thinking, working, asking any kinds of questions and freely expressing thoughts and ideas by everybody.

Pace

Some of my colleagues with whom I shared videos of Japanese lessons pointed to me that in many instances during the lesson they felt that the pace was slow. Interestingly, this is also the impression of many others (Stevenson & Stigler, p. 194). The slow pace of the lesson is related to its coherence (slowly building the "story" as described above) and to the teacher intention to be as inclusive as possible and to leave nobody behind. Moreover, this pace is a reflection of another deep belief: thinking takes time, ideas need to be mulled over, applied, discussed and approached from different directions, and if this is to be taken seriously there is no room for rushing.

Setting and devices

The 'architectural' setting of the classroom is traditional: lines of benches in rectangular rooms with a board at the front, simple teaching aids (such as magnetic manipulatives, paper cutting and the like). This setting does not prevent students from working with peers, come in groups to the board, and even moving around when the teacher thinks it is appropriate.

The blackboard plays a very central role in all classrooms, is not only a working space, but an organizational device, a thinking tool and a medium to record the flow of the lesson and its main ideas. Many times at the end of the class, if one looks at the board, it can tell the whole "story" of the lesson, especially displaying students' work and their differences of approaches.

Empathy

It is my impression that all students are affectively "contained" within a solid support net: teachers and parents work hard to closely follow up each of the students and attend to their needs as they arise. Teachers treat all their students with respect, they allow them to be boisterous and at times they even promote that. Some teachers also display their sense of humor, making the whole atmosphere of the class agreeable and supportive. Empathy seems to be yet another teaching strategy, which does not come at the expense of being intellectually demanding. Empathy seems to be characteristic of the way teachers address each other when discussing lesson plans and criticizing lessons. Teachers are used to expose their teaching to colleagues knowing that the analysis of their moves will be deep and thorough but very respectful and aimed at learning from each other.

The above characteristics are very different from each other. Some refer to a very deep pedagogical idea, some merely describe the physical setting, some present mathematical features, others refer to inter-human relationships. I would like to claim that maybe the uniqueness of the Japanese classrooms is due to the synergy of all these characteristics and to the professionalism with which of them each of them is treated.

Open questions

A first surprise regarding mathematics classrooms refers to something I did not see in them: computerized technologies. In spite of the many innovative proposals (and studies of their feasibility) about ways to introduce computerized and communication technologies both in the Japanese academia as well as in the academia of Western countries, I have not seen the use of computers in Japanese classrooms. In my view, the work with computerized technologies could fit in with the Japanese characterization of what mathematical thinking should be supported, and its availability in Japan is not an issue – thus, I wonder why it has not entered the classroom.

A second surprise refers to the shift in pedagogical practices that occur in secondary school, where most of the lessons consist of teacher lectures.

And finally, I wondered a lot about the existence, proliferation and success of “juku” schools (and possibly other out of school activities in mathematics), which are attended by a large number of students and are taken so much for granted by Japanese society at large, and which mostly emphasize drill and practice. This phenomenon can have several underlying reasons. For example, is it assumed and agreed that the time devoted by schools is insufficient, and drill should be learned elsewhere? Or, is it assumed that students need “extra practice”? Or, students should not have that much free time after school and their learning must be extended beyond the formal schooling? Or, students' full potential cannot be completely developed by the school only? Or, should students meet other teaching styles? Or….

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PLANNING A LESSON FOR STUDENTS TO DEVELOP MATHEMATICAL THINKING THROUGH PROBLEM SOLVING

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Teaching through problem solving has been emphasized in order to improve the teaching and learning of mathematics. However, it may not be easy for teachers to incorporate problem solving in their classrooms. An ideal way to incorporate problem solving is to plan a lesson and examine it through lesson study. This paper is intended to guide teachers in planning a lesson in which students will develop mathematical thinking through problem solving.

Developing mathematical thinking through problem solving

Teaching mathematics is for students to develop knowledge and skills that are mathematically important both for further study in mathematics and for use in applications in and outside of school is important for school mathematics. However, the objective of mathematics education is not only to enable students to acquire mathematical knowledge and skills but also to foster mathematical thinking. Mathematical thinking is crucial when students acquire and use mathematical knowledge and skills. In other words, students may have a difficult time acquiring and using knowledge and skills unless they have a sufficient ability to think mathematically.

In order to developing mathematical thinking, it is not enough for students simply to receive knowledge and skills by listening to teachers. Students need to actively engage in acquiring knowledge and skills, and to develop mathematical thinking through the process of mathematical activities. Thus students will be able to use these knowledge and skills effectively in their daily life as well as in their future carriers (Brown, 1994).

Based on the above assumption, it is suggested that teachers should provide students with opportunities to acquire knowledge and skills of mathematics through mathematical activities such as problem solving, reasoning and proof, communication, connection, and representation (National Council of Teachers of Mathematics, 2000). To implement such activity -based learning in mathematics classrooms, it is important for teachers to plan lessons that support students acquisition of the knowledge and skills by using mathematical thinking. Many teachers agree that teaching must emphasize the *process* of acquiring mathematics, But teachers often focus solely on teaching the contents to the students rather than providing students with opportunities to actually acquire the contents by using mathematical thinking. One of the reasons for teachers' hesitation to provide activities that cause students to develop mathematical thinking might be that the teachers have rarely experienced such lessons when they learned mathematics themselves. Moreover, planning lessons that focus more on students' learning process requires teachers to have more knowledge about their students, such as their thinking processes, in addition to having knowledge of the contents of mathematics (Simon & Tzur, 1999).

One of the ways to provide students with an opportunity to acquire not only knowledge and skills but also mathematical thinking is teaching mathematics through problem solving. Teaching mathematics through problem solving has been emphasized for decades, and many reform curriculum materials include problem solving as an integral part of learning mathematics (National Council of Teachers of Mathematics, 1980, 1989, 2000, 2006).

Problem solving in mathematics education is defined as "engaging in a task for which the solution method is not known in advance (National Council of Teachers of Mathematics, 2000)." This means that a problem suitable for problem solving is not necessarily a story problem or a problem in the real world. As long as a student does not know how to solve the problem, it can be a problem for problem solving for the student. In order word, even if a problem is presented as a real world story problem, it might not be a real problem for a student who already knows how to solve the problem. It is now called an exercise.

It is also important to note that teaching through problem solving is more than simply giving a task for students to solve a problem for which they have not learned the solution methods. Table 1. sho ws major differences between teaching of problem solving, which is a simplistic interpretation of the problem-solving approach, which can often be seen in traditional textbooks, and the teaching through problem solving, which is recommended by reform documents such as the NCTM Standards (2000).

Table 1. Problem-Solving Approach

Teaching of problem solving	Teaching through problem solving
• 'What is it? Problem Solving as an approach to develop problem-solving skills and strategies.	• 'What is it? Problem solving as a powerful approach for developing mathematical concepts and skills.
• How to incorporate it into a curriculum Usually the lessons based on this approach can be found at the end of chapters for developing problem-solving skills and strategies The lesson often end when each student comes up with a solution to the problem. (show and tell)	• How to incorporate it into a curriculum The lessons based on the approach can be found throughout the curriculum in order to develop mathematical concepts, skills, and procedures. The discussion for comparing students' different solutions is important for students to acquire new knowledge and understanding of mathematics.

Using lesson study to incorporate the idea of teaching through problem solving

Planning a lesson for lesson study is always challenging for teachers especially when teachers want to incorporate a new pedagogical idea that they have not experienced before. Lesson study is an ideal way to overcome such a challenging situation, because teachers can work together toward the same goal, which is to understand the new approach and to seek ways to incorporate it into each teacher's classroom (Takahashi & Yoshida, 2004).

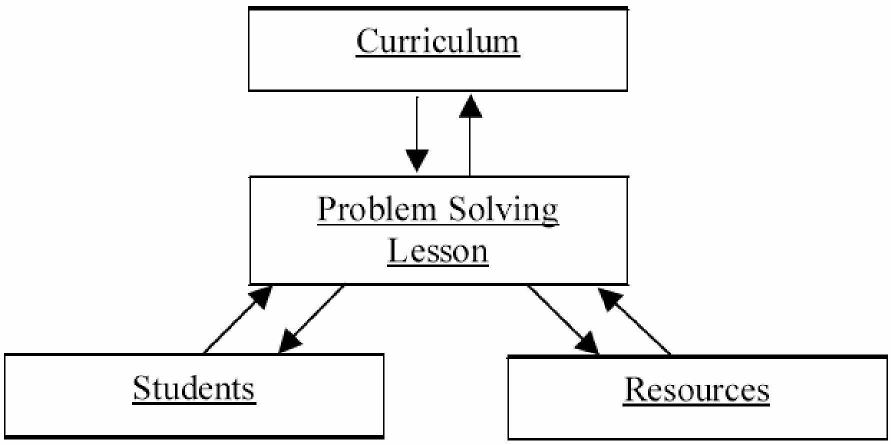
Although there are many different ways to plan a lesson for lesson study, it is often useful to examine each lesson carefully through the following three lenses; curriculum, students, and resources.

- Curriculum
  Any lessons that teachers prepare for their students must be purposeful and meaningful. Although there are many good problems for problem solving, it is important for teachers to identify what mathematics your students are expected to acquire through the lesson. This will become the goal of the lesson. In order to make the goal clear, teachers need to investigate the curriculum on which the students' mathematics learning is based – what they have already learned, what they are expected to learn now, and how their learning now will lead to their future learning. If the lesson is not fit into the curriculum well, it will not be helpful for students to accomplish the goal of the lesson.
- Students
  Knowing students is crucial when teachers plan a lesson. Especially for the lessons that are designed for students to acquire mathematical thinking, teachers need to know how the students might attempt to solve the problem. Without anticipating their students' approaches to the problem, teachers may not be able to plan how to lead the students to develop mathematical thinking by using their informal approaches to solving the problem. It is also important to anticipate students' typical misunderstandings so that teachers can be ready to support the students in overcoming their misunderstandings.
- Resources
  Choosing the best resources is also an important part of planning lessons. These resources include not only good problems in textbooks and other resource materials but also manipulatives, video, and interactive tools on the internet. Moreover, simply knowing what resources are available is not

enough. Teachers should know the potential benefits and drawbacks of each resource. For example, there are problems that are interesting and fun but that may not lead students to develop mathematical thinking at a particular time. When examing the lesson through this lens, teachers might want ask themselves if this is the best resource for students to reach the goal.

In Japanese, the above process of investigating the curriculum, students, and resources is called *Kyozai Kenkyu.* This investigation is important groundwork for planning lessons. The quality of each lesson will greatly rely upon the deepness of *Kyozai Kenkyu. Kyozai Kenkyu* becomes extremely important when teachers plan lessons to address new teaching agenda such as developing mathematical thinking through problem solving.

Figure 1. Planning a Problem Solving Lesson



Planning a lesson for students to develop Mathematical Thinking through Problem Solving

When begin to plan a problem-solving lesson, there are multiple entry points. For example teachers might want to begin *Kyozai Kenkyu* by carefully examining the curriculum to identify what mathematics the students are expected to acquire. Another entry point is to begin with examining students' work to identify what might be the area where students need to deepen their understanding in order to improve their mathematical ability. The third entry point is by examining resources to plan a problem-solving lesson. Although planning lessons in all three ways is encouraged for teachers, it is important to closely look at the lesson planning process by using the third entry point, resources, because many novice teachers take a wrong pass after they chose an attractive problem.

When teachers read teacher resource books and textbooks, and participate in professional development workshops and conferences, they often find an interesting and fun problem for students. Although these problems are useful resources, it is important to note that they are raw material and need to be prepared for a problem-solving lesson. It is not a good idea for teachers to bring those raw materials to classrooms to simply ask students to solve them or to show students how to solve problems. When teachers do not have a clear goal of the lesson, the lesson often become meaningless for the students.

At the APEC Tokyo/Sapporo symposia, Stacey (2006) used an interesting website, "Crystal Ball[1]", to illustrate the processes of mathematical thinking in the context of a problem solving lesson. If teachers are inspired by her talk and want to use the website to plan a problem-solving lesson for their own students to develop mathematical thinking, what should they do?

Investigating the problem

As Stacey (2006) describes, there are several ways to find out the trick. It is important for teachers to attempt several different approaches to discover mathematics behind the "Crystal Ball". In this particular case, teachers want to spend time to find the trick by themselves. Then, they should look at the same web page from students' viewpoints, asking themselves how could they find out the trick if they were a fifth grade student or a seventh grade student? This will lead teachers to investigate the problem through the student lens, although the investigation originally began through resource lens. After trying to figure out find out the trick in various ways, it might be a good idea to compare all the approaches for figuring out the trick to see how these approaches are related and how they are different. What mathematical knowledge and skills, and mathematical thinking are required for each approach?

Through this investigation, a group of teachers might be able to come up the following conclusion.

1 http://www.cyberglass.biz/customflash/ghostwhisperer/

In order to find out the trick, one of the approaches is to try several specific examples to find a pattern among the examples. Students typically use this inductive approach and find out that there might be mechanism behind the trick, but it is difficult to figure out why the pattern exists. Another approach is to investigate the process of calculations described in the “Crystal Ball” instruction in order to find out what calculations are actually carried out to get the symbol that you need to imagine. This deductive approach demands that students write, interpret, and use mathematical expressions to investigate the trick, then find out why the crystal ball gives you the same symbol no matter what two digit numbers are chosen. During this investigation, students will be using their previous learning of the properties of the basic operations, the notion of place value, and the use of symbols in mathematical expressions to see the generalized pattern.

Investigating the problem through other lenses

The next step toward planning a lesson by using the “Crystal Ball” might be to narrow down what mathematics teachers expect students to acquire through this problem solving. From previous investigation, teachers agree that most students should be able to try at least a couple of specific cases to draw a conclusion that there might be a trick behind the website. Moreover, some of the students might be able to find that the procedure that the “Crystal Ball” gives always produces a number that is a multiple of nine. It is, however, expected that many students might not be able to figure out why the procedure always gives a number that is multiple of nine, because it requires students to manipulate mathematical expressions.

The above process describes how teachers can investigate the problem through the lens of "students". The next step might be the investigation through the "curriculum" lens. In the *Curriculum focal points for pre -kindergarten through grade 8 mathematics: a quest for coherence* (National Council of Teachers of Mathematics, 2006), one of the focal points in the middle school is to write, interpret, and use mathematical expressions and equations to solve problems. It is expected that students become able to

1) write mathematical expressions and equations that correspond to given situations,
2) evaluate expressions, and
3) use expressions and formulas to solve problems.

One of the challenges for the students is to write mathematical expressions that correspond to a given situation. Sometimes students may be reluctant to write mathematical expressions because they often try to find the answer by simply carrying out calculations and cannot see the merits of writing mathematical expressions. In order to overcome students’ reluctance to write mathematical expressions, therefore, it is important that they learn how writing mathematical expressions that can help them to solve problems.

Form that discussion, the goal of this problem-solving lesson might be to provide students an opportunity to learn inductive reasoning by writing, interpreting, and using mathematical expressions.

Designing the flow of the lesson

After going through the ground work, *Kyozai Kenkyu,* the group of teachers should move toward actually discussing how to pose the problem, and what questions a teacher can ask the students toward acquiring mathematical knowledge and skills. There are several types of lesson plans for lesson study. One important component that most Japanese lesson plans share is the section called "anticipated students' responses." The quality of the section of student anticipated responses relies heavily on the richness of *Kyozai Kenkyu*. Moreover, this section usually contributes greatly to the quality of the discussion that a teacher will be leading after students present their various solutions.

It is often a good idea for teachers to prepare answers for the following questions to develop a short sketch of the lesson.

- Purpose of the problem solving (goal of the lesson)

  What mathematics, beside developing problem solving skills, would you teach by using this situation?

- Questioning

  How would you pose the problem?
  What question(s) would you ask of your students for them to learn mathematics?

- Beyond show and tell

  Anticipate students' responses to your questions, including misunderstandings, to facilitate discussion.
  Briefly describe how you would facilitate discussion.

The Appendix shows an example of the lesson plan for the problem-solving lesson using the "Crystal Ball."

Conclusion

Planning lessons for lesson study demands that teachers spend time and effort. Although it is time consuming, once teachers experience this process with their colleagues, they start seeing their everyday lessons differently. It is not easy for teachers to change their teaching practice in a short time. However, it will be a great step toward addressing mathematical thinking in their everyday lessons.

It will be also a powerful experience for teachers to observe an actual classroom based on the lesson plan that the group planned together. Since the mathematical thinking can be observed mostly in the process of students' problem solving and dialogues among students, the entire process of lesson study is expected to improve mathematical thinking.

1 http://www.cyberglass.biz/customflash/ghostwhisperer/

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8TH GRADE MATHEMATICS LESSON PLAN

April 26, 2007
Las Cruces, NM
Instructor: Akihiko Takahashi

1. Title of the lesson: The Secret of The Crystal Ball
2. Goal of the lesson:
    1. To deepen students' understanding of the properties of the basic operations and place value by writing, interpreting, and using mathematical expressions through problem solving
    2. To help students become good problem solvers by
        - i. encouraging them to use their prior knowledge to examine a problem situation in order to develop their ability to use logical reasoning to make conjectures, and
        - ii. encouraging them to examine and justify the conjectures presented by their peers in order to find a solution to the problem.
    3. Provide opportunities for students to recognize the importance of working with their peers in order to deepen their understanding of mathematics

3. Instruction of the Lessons

In the Curriculum focal points for pre-kindergarten through grade 8 mathematics: a quest for coherence (National Council of Teachers of Mathematics Inc. Reston VA., 2006), one of the
focal points in the middle school is to write, interpret, and use mathematical expressions and equations to solve problems. It is expected that students become able to:

1) write mathematical expressions and equations that correspond to given situations,
2) evaluate expressions, and
3) use expressions and formulas to solve problems.

One of the challenges for the students is to write mathematical expressions that correspond to a given situation. Sometimes students may be reluctant to write mathematical expressions because they often try to find the answer by simply carrying out calculations and cannot see the merits of writing mathematical expressions. In order to overcome students' reluctance to write mathematical expressions, therefore, it is important that they learn how writing mathematical expressions can help them to solve problems.

When designing such problem-solving lesson, it is important to keep in mind that solving a problem is a process for providing an opportunity for students to appreciate that writing, interpreting, and using mathematical expressions. Therefore, the flow of the lesson should not solely focus on finding the correct answer, but also the process of solving the problem.

This lesson is designed for students' to understand how writing, interpreting, and using mathematical expressions help them analyze the problem situation and empower them to solve a problem.

The problem for this lesson to figure out the mechanism behind a trick named "Crystal Ball" from the website of a popular TV program, Ghost Whisperer (http://www.cbs.com/primetime/ghost _whisperer/crystal b all.shtml). The website is based on a popular math trick and use Flash, multimedia authoring program for web applications, to make it interactive and engaging. The procedures that described on the website is

> *Chose any two digit number, add together both digits and then subtract the total from your original number. When you have the final number look it up on the chart and find the relevant symbol. Concentrate on the symbol and when you have it clearly in your mind click on the Ghost Whisperer crystal ball and it will show you the symbol you are thinking of*

In order to find out the trick, one of the approaches is to try several specific examples to find a pattern among the examples. Students typically use this inductive approach and find out that there might be mechanism behind the trick but it is difficult to figure out why the pattern exists. Another approach is to investigate the process of calculations described in the "Crystal Ball" instruction in order to find out what calculations are actually carried out to get the symbol that you need to imagine. This deductive approach demands that students write, interpret, and use mathematical expressions to investigate the trick, then find out why the crystal ball always gives you the same symbol no matter what two digit numbers are chosen. During this investigation, students will be using their previous learning of the properties of the basic operations, the notion of place value, and the use of symbols in mathematical expressions to see the generalized pattern.

4) Flow of the Lesson

Learning Activities, Teacher's Questions and Expected Students' Reactions	Teacher's Support	Points of Evaluation
1. Introduction to the Problem By experiencing the "Crystal Ball" on the internet, students will become familiar with the site. 1. Chose any two digit number, 2. Add together both digits, 3. Subtract the total from your original number 4. When you have the final number look it up on the chart and find the relevant symbol. 5. Concentrate on the symbol and when you have it clearly in your mind 6. Click on the crystal ball to see the symbol	Ask a couple of volunteer students to try the website so that all the students understand the procedures described on the webpage. Help students to see the website always gives you the relevant symbol.	Do students understand the procedure? Do students see what is happening on the website?
2. Posing the problem By asking the following question, engage students to find the trick behind the "Crystal Ball" webpage. With which opinion do you agree? a. It is just a coincident and there is nothing special in the "Crystal Ball" webpage. b. There might be a trick behind the "Crystal Ball". c. The "Crystal Ball" webpage actually reads your mind. Let's find the trick behind the "Crystal Ball" webpage!	Each student will be working with his/her partner to find a trick by using their prior knowledge. Provide students with worksheets to keep their work for the whole class discussion.	Do students see there must be a trick behind the "Crystal Ball" webpage

3. Problem Solving Working with a partner, students try to find the trick behind the "Crystal Ball" webpage. Anticipated students' responses: a. Try a couple of specific examples to notice that the relevant symbol might be always the same but do not know why these symbols are the same. b. By examining several specific examples, he/she realizes that the final number will always be a multiple of nine, and the symbols on the chart that correspond to multiple of nine are all the same. However, he/she does not know why the final number will always be a multiple of nine. c. Write, interpret, and use mathematical expressions to investigate the trick - ab as a chosen two digit number - The value of ab is $10a + b$ - Write a mathematical expression to express the procedure $(10a + b)(a + b)$ $= 10a + b a b$ $= 10aa + bb$ $= 9a$ Therefore the final number will always be a multiple of nine	Encourage students to try at least a couple of specific examples. Help students understand that methods (a) and (b) may not be able to answer the question why all the final numbers give you the same symbol. Encourage students to investigate the process of calculations described in the instructions to "Crystal Ball" in order to find out what calculations are actually carried out to get the symbol that you need to imagine.	Do students try at least a couple of specific examples to notice that the relevant symbol from your calculation might always be the same.
4. Discussing Students' Solutions (1) Ask students to explain their solutions to the other students in the class. (2) Facilitate students' discussion about their solutions, then lead them to understand that writing, interpreting, and using mathematical expressions helped them understand the trick behind the "Crystal Ball" webpage.	Write students' solutions and ideas on the blackboard in order to help students understand the discussion.	Can students explain their solutions to their peers? Can students examine and justify the solutions presented by their peers?
5. Summing up (1) Using the writing on the blackboard, review what students learned through the lesson. (2) Ask students to write a journal entry about what they learned through this lesson.	Encourage students to use the writing on the board as a reference when they write the journal entry.	

Reference

National Council of Teachers of Mathematics Inc. Reston VA. (2006). *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence.*

http://www.cbs.com/primetime/ghost whisperer/crystal ball .shtml

Board writing Plan for the "Crystal Ball"

The Crystal Ball

1. Chose any two digit number,
2. Add together both digits,
3. Subtract the total from your original number
4. When you have the final number look it up on the chart and find the relevant symbol
5. Concentrate on the symbol and when you have it clearly in your mind click on the crystal ball to see the symbol

What is happening on the website?
Use the worksheet to figure out

Case 1: 56
5+6=11
56-11=45 Symbol A

Case 2: 78
7+8=15
78-15=63 Symbol A

Students' approach A

Students' approach B

Students' approach C

The Crystal Ball always give you the same symbol no matter what two digit numbers are chosen because

1. the final number that the procedure give by the Crystal Ball always be a multiple of 9,
2. the symbols on the chart that correspond to multiple of 9 are all the same. (with two exceptions, 90 and 99)

SETTING LESSON STUDY WITHIN A LONG-TERM FRAMEWORK OF LEARNING

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Lesson Study is a format to build and analyse classroom teaching where teachers and researchers combine to design lessons, predict how the lessons might be expected to develop, then carry out the lessons with a group of observers bringing multiple perspectives on what actually happened during the lesson. This article considers how a lesson, or group of lessons, observed as part of a lesson study may be placed in a long-term framework of learning, focusing on the essential objective of improving the long-term learning of every individual in classroom teaching.

INTRODUCTION

This paper began as a result of a participation in a lesson study conference (Tokyo & Sapporo, December 2006) in which four lessons were studied as part of an APEC (Asian and Pacific Economic Community) study to share ideas in teaching and learning mathematics to improve the learning of mathematics throughout the communities. It included the observation of four classes (here given in order of grade, rather than order of presentation):

> Placing Plates (Grade 2)
> December 2nd 2006, University of Tsukuba Elementary School
> - Takao Seiyama
>
> Multiplication Algorithm (Grade 3)
> December 5th 2006, Sapporo City Maruyama Elementary School
> - Hideyuki Muramoto
>
> Area of a Circle (Grade 5)
> December 2nd 2006, Universty of Tsukuba Elementary School
> - Yasuhiro Hosomizu
>
> Thinking Systematically (Grade 6)
> December 6th 2006, Sapporo City Hokuto Elementary School
> - Atsutomo Morii

The objective of this paper is to set these classes within a long -term framework of development outlined in Tokyo at the conference (Tall, 2006), which sets the growth of individual children within a broader framework of mathematical development. Long-term the development of individual children depends not only on the experiences of the lesson, but in the experiences of the children prior to the lesson and how experiences ‘met-before’ have been integrated into their current knowledge framework.

In general, it is clear that lesson study makes a genuine attempt:

to design a sequence of lessons according to well-considered objectives;
to predict what may happen in a lesson;

to have a group of observers bring multiple perspectives to what happened, without prejudice; and ultimately

to improve the teaching of mathematics for all.

Lesson study is based on a wide range of communal sharing of objectives. At the meeting I was impressed by one essential fact voiced by Patsy Wang -Iverson:

> The top eight countries in the most recent TIMMS studies shared a single charactistic, that they had a smaller number of topics studied each year.

Success comes from focusing on the most generative ideas, not from covering detail again and again. This suggests to me that we need to seek the generative ideas that are at the root of more powerful learning.

For many individuals, mathematics is *complicated* and it gets more complicated as new ideas are encountered. For a few others, who seem to grasp the essence of the ideas, the *complexity* of mathematics is fitted together in a way that makes it essentially *simple* way. My head of department at Warwick University in the sixties, Sir Christopher Zeeman noted perceptively:

> "Technical skill is a mastery of complexity, while creativity is a mastery of simplicity" (Zeeman, 1977)

This leads to the fundamental question:

> How can we help *each and every child* find this simplicity, in a way that works, *for them?*

Lesson study focuses on the *whole class activity*. Yet within any class each child brings differing levels of knowledge into that class, related not only to what they have experienced before, but how they have made connections between the ideas and how they have found their own level of simplicity in being able to think about what they know.

To see simplicity in the complication of detail requires the making of connections between ideas and focusing on essentials in such a way that these simple essentials become generating principles for the whole structure.

In my APEC presentation in Tokyo (Tall 2006), I sought this simplicity in the way that we humans naturally develop mathematical ideas supported by the shared experiences of previous generations. I presented a framework with three distinct worlds of mathematical development, two of which dominate development in school and the third evolves to be the formal framework of mathematical research. The two encountered in school are based on (conceptual) embodiment and (proceptual) symbolism. I described these technical terms in more detail in Tall (2006) and in a range of other recent papers on my website (www.davidtall.com/papers).

Essentially, conceptual embodiment is based on human perception and reflection. It is a way of interacting with the physical world and perceiving the properties of objects and, through thought experiments, to see the essence of these properties and begin to verbalise them and organize them into coherent structures such as Euclidean geometry. Proceptual symbolism arises first from our *actions* on objects (such as counting, combining, taking away etc) that are symbolized as concepts (such as number) and developed into symbolic structures of calculation and symbolic manipulation through various stages of arithmetic, algebra, symbolic calculus, and so on. Here symbols such as 4+3, $x^2 + 2x + 1$, ! sin x dx all dually represent processes to be carried out (addition, evaluation, integration, etc) and the related concepts that are constructed (sum, expression, integral, etc). Such symbols also may be represented in different ways, for instance 4+3 is the same as 3+4 or even '1 less than 4+4' which is '1 less than 8' which is 7. This flexible use of symbols to represent different *pro*cesses for giving the same underlying con*cept* is called a *procept.*

These two worlds of (conceptual) embodiment and (proceptual) symbolism develop in parallel throughout school mathematics and provide a long-term framework for the development of mathematical ideas throughout school and on to university, where the

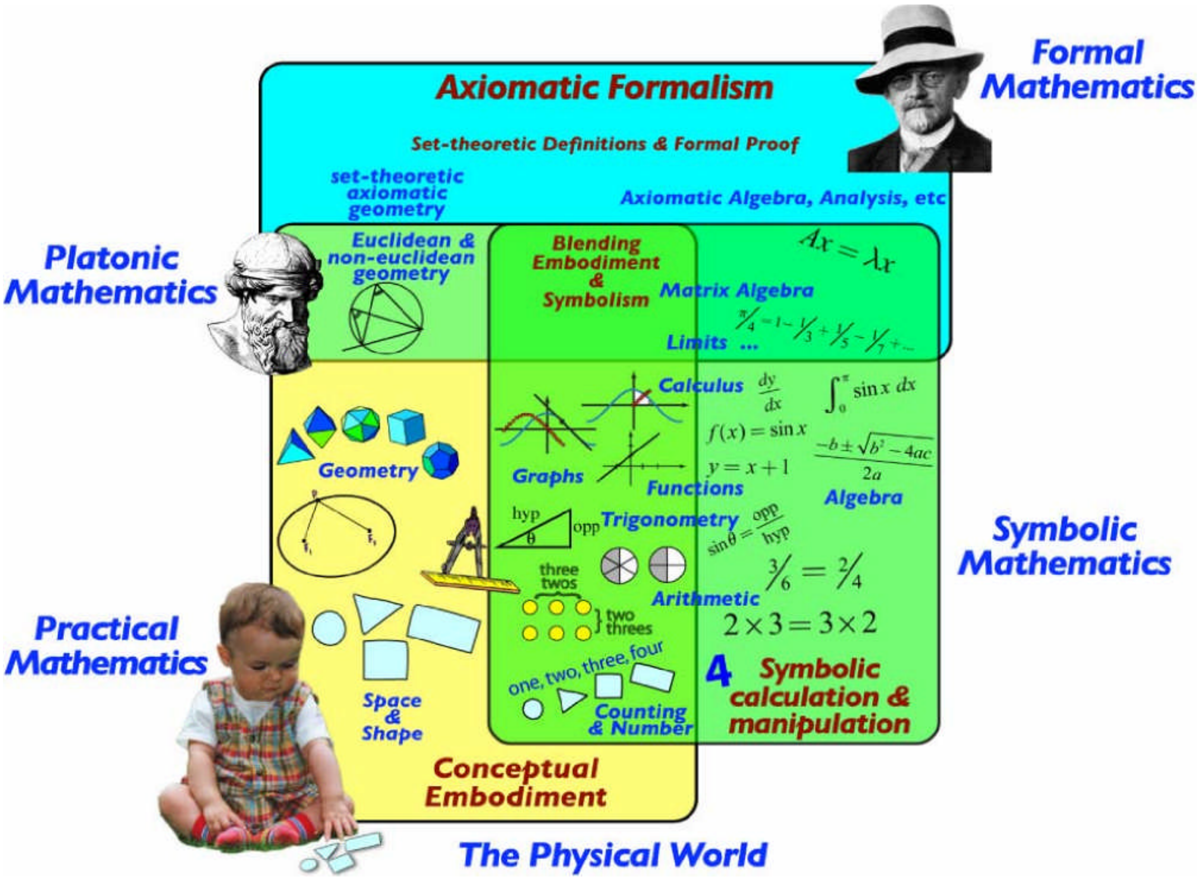


Figure 1. The three mental worlds of (conceptual) embodiment, (proceptual) symbolism and (axiomatic) formalism

focus changes to the formal world of set-theoretic definition and formal proof.

In figure 1 we see an outline of the huge *complication* of school mathematics. On the left is the development of conceptual embodiment from practical mathematics of physical shapes to the platonic methods of Euclidean geometry. In parallel, there is a development of symbolic mathematics through arithmetic, algebra, and so on, with the two blending as embodiment is symbolized or symbolism is embodied.

The long-term development begins with the child's perceptions and actions on the physical world. In figure 1 the child is playing with a collection of objects: a circle, a triangle, a square, and a rectangle. The child has two distinct options, one to focus on his or her *perception* of each object, seeing and feeling their separate properties, the other is through *action* on the objects, say by counting them: one, two, three, four.

The focus on perception, with vision assisted by touch and other senses to play with the objects to discover their properties, leads to a growing sense of space and shape, developing through the use of physical tools—ruler, compass, drawing pins, thread—to enable the child to explore geometric ideas in two and three dimensions, and on to the mental construction of a perfect platonic world of Euclidean geometry. The focus on the essential qualities of points having location but no size, straight lines having no width but arbitrary extensions and on to figures made up using these qualities leads the human mind to construct mental entities with these essential properties. Platonism is a natural long-term construction of the enquiring human mind.

Meanwhile, the focus on action, through counting, leads eventually to the concept of number and the properties of arithmetic that benefit from blending embodiment and symbolism, for example, 'seeing' that 2 x 3= 3 ! 2 by visualizing 2 rows of 3 objects being the same as 3 columns of 2 objects. Long-term there is a development of successive number systems, fractions, rationals, decimals, infinite decimals, real numbers, complex numbers. (What seems to the experienced mathematician as a steady extension of number systems is, for the growing child, a succession of changes of meaning which need to be addressed in teaching. We return to this later.)

The symbolic world develops through whole number arithmetic, fractions, decimals, algebra, functions, symbolic calculus, and so on, which are given an embodied meaning through the number-line, Cartesian coordinates, graphs, visual calculus, with aspects of the embodied world such as trigonometry being realized in symbolic form. In the latter stages of secondary schooling, the learner will meet more sophisticated concepts, such as symbolic matrix algebra and the introduction of the limit concept, again represented in both embodied and symbolic form.

The fundamental change to the formal mathematics of Hilbert leads to an axiomatic formalism based on set-theoretic definitions and formal proof, including axiomatic geometry, axiomatic algebra, analysis, topology, etc.

Cognitive development works in different ways in embodiment, symbolism and formalism (Figure 2). In the embodied world, the child is relating and operating with perceived objects (both specific and generic), verbalizing properties and shifting from practical mathematics to the platonic mathematics of axioms, definitions and proofs.

In the symbolic world, development begins with actions that are symbolized and coordinated for calculation and manipulation in successively more sophisticated contexts. The shift to the axiomatic formal world is signified by the switch from concepts that arise from perceptions of, and actions on, objects in the physical world to the verbalizing of axiomatic properties to define formal structures whose further properties are deduced through mathematical proof.

Focusing on the framework appropriate to school mathematics, we find the main structure consists of two parallel tracks, in embodiment and symbolism, each building on previous experience (met-befores), with

> *embodiment* developing through perception, description, construction, definition, deduction and Euclidean proof after the broad style suggested by van Hiele;
>
> *symbolism* developing through increasingly sophisticated compression of procedures into procepts as thinkable contexts operating in successively broader contexts.

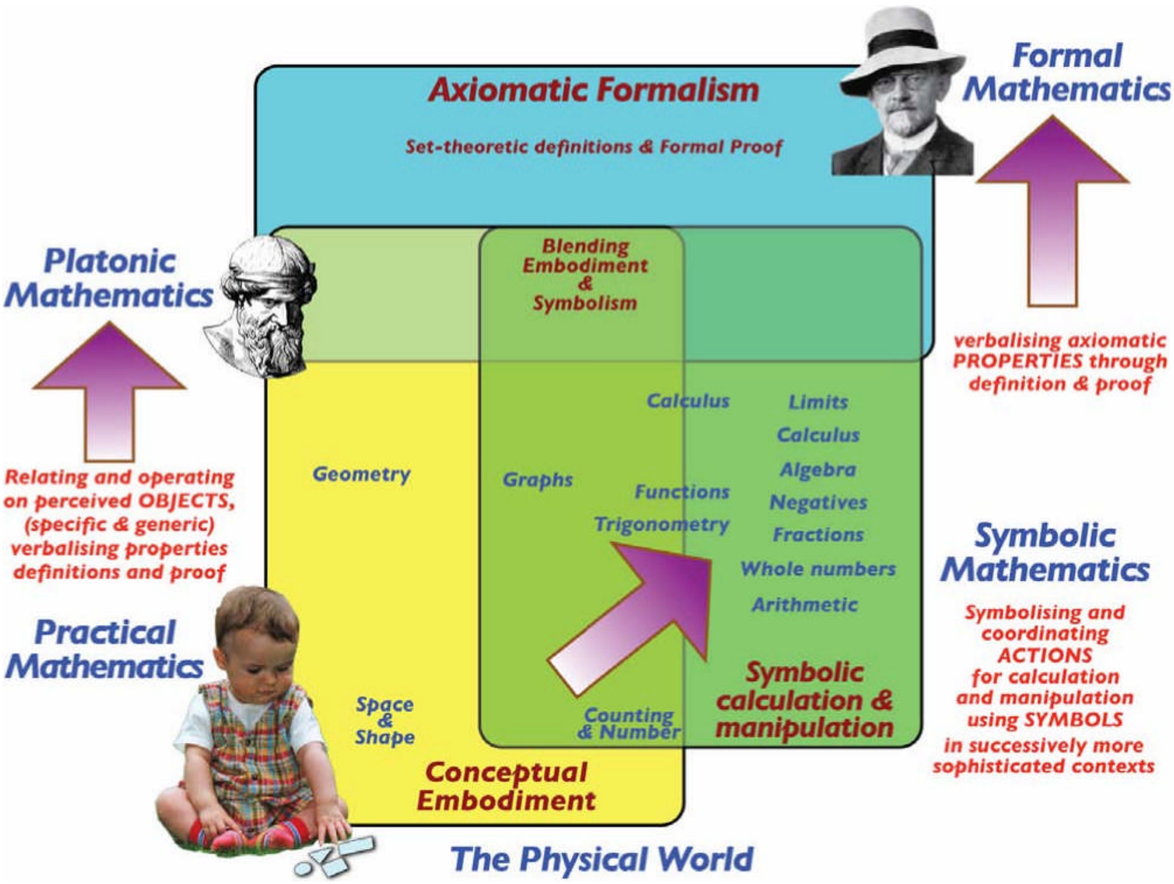


Figure 2: long-term developments in the three worlds

These two developments are fundamentally different. On the one hand, embodiment gives a global overall picture of a situation while symbolism begins with coordinating actions, practicing sequences of actions one after another to build up a procedure, perhaps refining this to give different procedures that are more efficient or more effective, using symbolism to record the actions as thinkable concepts. The problem here is that the many different procedures can, for some, seem highly complicated and so the teacher faces the problem of reducing the complexity, perhaps by concentrating on a single procedure to show the pupils what to do, without becoming too involved in the apparent complications. Procedures, however, occur *in time* and become routinized so that the learner can *perform* them, but is less able to *think about* them. (Figure 3.)

As an example, consider the teaching of long-multiplication. First children need to learn their tables for single digit multiplication from 0 ! 0 to 9 ! 9. They also need to have insight into place value and decimal notation.

The method used by Hideyuki Muramoto in the lesson study at Sapporo City Maruyama Elementary School on December 6, 2006 can be analysed in terms of an initial embodiment representing 3 rows of 23. Here the learner can *see* the full set of counters: the problem is how to *calculate* the total. The embodiment can be broken down in various ways, separating each row into subsets appropriate to be able to compute the total. In the previous lesson the students had already considered 3 rows of 20 and had broken this into various sub-combinations, breaking each row into 10+10 or 5+5+5+5, or even 9+9+2, or 9+2+9. Now the problem related to breaking

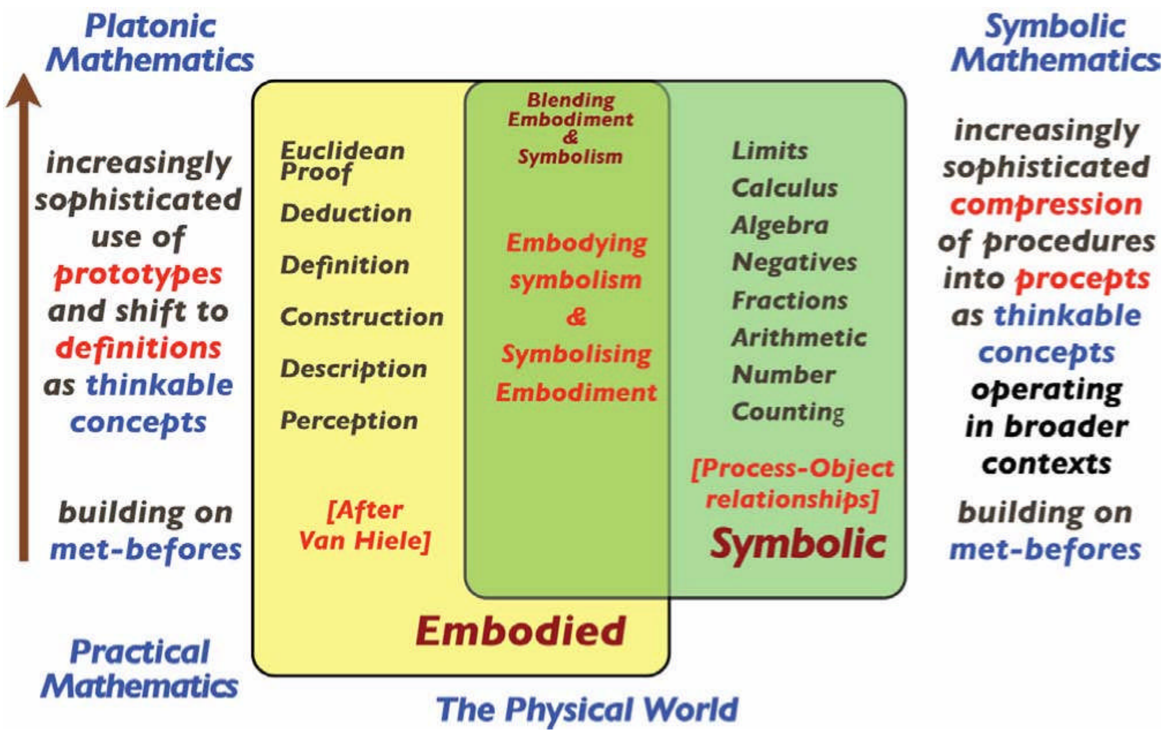


Figure 3: Developmental framework through embodiment and symbolism

23 into sub-combinations, suggested possibilities included 10+10+3 and 9+5+9 (but not 5+5+5+5). Three lots of 10+10+3 gives 30+30+9, which easily gives 60+9, which is 69. Three lots of 9+5+9 is more difficult requiring the sum 27+15+27. Here we have two different procedures giving the same result, 69, and the most productive way forward is to break the number 23 into tens and units and multiplying each separately by 3.

In this analysis, the embodiment gives the *meaning* of the calculation of a single digit times a double digit number, while the various distinct sub-combinations give different ways of *calculation*, from which the sub-combination as tens and units is clearly the simplest and the most efficient.

The approach has a general format:

1. *Embody the problem* (here the product 23 ! 3);
2. *Find several different ways of calculation* (here 23 ! 3 is three lots of 10+10+3 or three lots of 9+9+5) *where the embodiment gives meaning to symbolism;*
3. See *flexibility*, that all of these are the same;
4. See the *standard algorithm is the most efficient.*

Thus embodiment gives meaning while symbolism enables compression to an efficient symbolic algorithm.

It may be that not all the children in the class will be able to cope with the different procedures (for instance, one would expect the suggestion 9+5+9 to come from a more able child and the computation would not be easy for some). Thus, the dynamic of the whole class may not be shared by all individuals. The more successful may see the different ways of computing the result as different procedures with the same

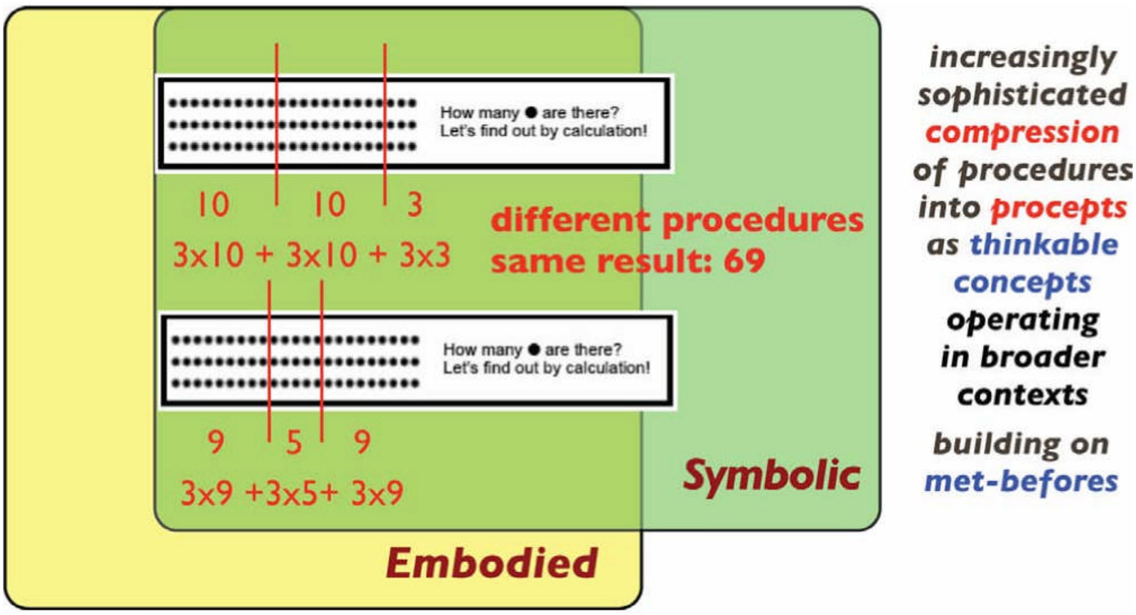


Figure 4: multi-digit arithmetic from embodiment to symbolism

effect, and meaningfully see that the standard algorithm is just one of many that is chosen because it is efficient and simple. They may sense that it is not appropriate to use a more complicated method like 3 times 9+9+5 and not even desire to carry it through without this compromising their insight that different procedure s can give the same result. Meanwhile, those who are less fluent in their tables may feel insecure and seek an easy method to cope that is less complicated. A single procedure may have its attractions, showing *how to do it,* without the complication of *why it works.* It may have attractions to the teacher to teach the method by rote as this may have short-term success without extra complication.

In this way, the same lesson may be seen very differently by different participants, at one extreme, a great insight into the meaning and construction of the standard algorithm within a rich conceptual framework, at another extreme, a great deal of complication and a desire to cope by seeking a procedure *that works* rather than a situation which is too complicated to understand. This bifurcation is what Gray & Tall (1994) called the *proceptual divide* between those who seek to maintain procedures that work at the time rather than flexible methods that require many meaningful connections in a broader knowledge structure.

BLENDING KNOWLEDGE STRUCTURES IN THE BRAIN

In addition to this combination of embodiment and symbolism to give meaning to number concepts and operations, there are subtle features of successive number systems that cause additional problems. A mathematician may see successive numbers systems such as:

Whole Numbers

Fractions

Rational Numbers

Positive and Negative numbers

Real Numbers consisting of rationals and irrationals

as a growing extension of the number system. They can all be marked on an (embodied) number line and the child should be able to *see* how each one is extended to the next. However, for the learner, each extension has subtle aspects which can cause significant problems. We all know of the difficulty of introducing the concept of fraction and of the problem of multiplying negative numbers. There are subtle difficulties between counting and measuring:

Counting 1, 2, 3, ... has successive numbers, each with a next number and no numbers in between. Multiplying these numbers gives a bigger result ... etc.

Measuring numbers are continuous without a ‘next’ number and have fractions between. Multiplying can give a smaller result.

Elsewhere (e.g. Tall, 2007), I use the idea of *conceptual blending* from Fauconnier & Turner (2003) to shed light onto the cognitive strengths and difficulties of long -term learning in mathematics. Fauconnier and Turner share the distinction of being the first cognitive scientists to integrate the fundamental ideas of *compression* and *blending* of knowledge into a single framework. In considering how students learn long-term, this suggests we need to be aware not only what experiences students have had before, but how they compress this experience into thinkable concepts and how different knowledge structures are blended together to produce new knowledge.

USING A LONG-TERM FRAMEWORK OF EMBODIMENT AND SYMBOLISM IN LESSON STUDY

Putting together the ideas of growth in elementary mathematics discussed here and in the earlier paper (Tall, 2006), we find that the parallel development of embodiment and symbolism suggests:

> Embodiment gives human meaning as prototypes, developing verbal description, definition, deduction.
>
> Symbolism is based initially on human action, leading to symbol use, either through procedural learning or through conceptual compression to flexible procept.
>
> Experiences build met-befores in the individual mind that are used later to interpret new situations.
>
> Different experiences may be blended together, requiring a study of what learners bring to a new learning experience.

Tall (2006) also observed:

> Embodiments may work well in one context but become increasingly complex; flexible symbolism may extend more easily.

This means that successful students may show a long-term tendency to shift to symbolism to work in a way that is both more powerful and (for them) more simple.

In our earlier discussions in Tokyo, great emphasis was made not only on meaningful learning of mathematical concepts and techniques, but also on *problem-solving* in new contexts. Learning new concepts can be approached in a problem-solving manner. My own view is that learners must take responsibility for their own learning, once they have the maturity to do so, which includes developing their own methods for solving problems. I also believe that teachers have a duty, as mentors, to help focus students on methods that are powerful and have long-term value.

In studying lessons, therefore, we need some objectives to consider. There are so many theories in the literature, from Bruner's (1966) analysis into enactive iconic and symbolic, Fischbein's (1987) categorization into intuitive, algorithmic and formal, the Pirie-Kieren theory (1994) with its ideas of 'making' and 'having' images and successive levels of operation, Dreyfus and colleagues RBC theory (Recognising, Building-With, Consolidating), theories of problem-solving (Schoenfeld 1985, Mason *et al.* 1982) and so on. With such a wealth of ideas to choose from and build on (and build with), I will hear focus on three simple ideas that are important. You may choose different ones, but in the long run, it is important for those studying lessons to have principles with which they are working and a fundamental framework for each lesson study. I suggest the need in long-term development to focus on three aspects:

Building thinkable concepts in (*meaningful)* knowledge structures;

Using knowledge structures in *routine* and *problem* situations (where 'routine' includes practising for fluency);

Proving knowledge structures (*as required in context).*

I would see these three aspects being applied *before, during* and *after* each lesson.

BEFORE: What is the purpose of the lesson

(e.g. **Building** new constructs, **Using** *known routines* or *problem-solving*, **Proving** in some sense) and what concepts may the learners have in mind that may be used in the lesson? (*met-befores, blends, routines, problem-solving techniques)*

DURING: How do learners use their knowledge structures during the lesson to make sense of it? (*met-befores, blends, routines, problem-solving techniques)*

AFTER: What knowledge structures are developing that may be of value in the future? (*met-befores, blends, routines, problem-solving techniques)*

LESSONS STUDIES

Four classes were videoed during our previous meeting in Japan, December, 2006.

Placing Plates (Grade 2)
December 2nd 2006, Universty of Tsukuba Elementary School
- Takao Seiyama;

Multiplication Algorithm (Grade 3)
December 5th 2006, Sapporo City Maruyama Elementary School
- Hideyuki Muramoto;

Area of a Circle (Grade 5)
December 2nd 2006, University of Tsukuba Elementary School
- Yasuhiro Hosomizu;

Thinking Systematically (Grade 6)
December 6th 2006, Sapporo City Hokuto Elementary School
- Atsutomo Morii.

My purpose is to focus on the role of these lessons in long-term learning, and to consider how the long-term development of each and every student may be affected by the lesson within the framework suggested above.

There is already a great deal of evidence of the use of broad principles in the planning of the lessons which are formulated in the lesson plans. Taking a few quotes at random we find:

> The goal of the Mathematics Group at Maruyama is to develop students ability to use what they learned before to solve problems in the new learning situations by making connections.
>
> In addition, we want to provide 3rd grade students with experiences in mathematics that enable them to use why they learned before to give problems in new learning situations by making connections.
>
> Through teaching mathematics, I would like my students to develop 'secure ability' for finding problems on their own, studying by themselves, thinking, making decisions, and executing those decisions. Moreover, I would like to help my students like mathematics as well as enjoy thinking.
>
> In order for students to find better ideas to solve the problem, it is important for the students to have an opportunity to feel that they really want to do so.
>
> Starting in April (beginning of the school year), I taught the students to look at something from a particular point of view such as 'faster, easier, and accurate' when they think about something or when they compare something.
>
> If you think about the method that uses the table form this point of view, students might notice that "it is accurate but it takes a long time to figure out: or "it is accurate but it is complicated."
>
> In order to solve a problem in a short time and with less complexity, it is important for the students to notice that calculation using a math sentence in necessary.

Each of these shows a genuine desire for students to make connections, to rely on themselves for making decisions and to seek more powerful ways of thinking with less complexity. The videos of the classes themselves show high interaction between the students, and with the teacher, carefully orchestrated by the teacher to bring out essential ideas in the lesson.

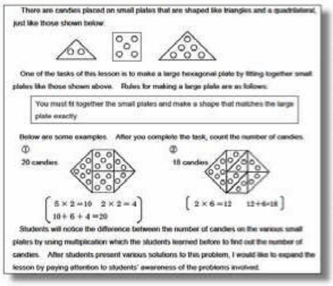
We now briefly look at each lesson in turn, to see how it fits with a long-term development blending embodiment and symbolism, what aspects of Building, Using, and Proving arise as an explicit focus of attention, before, during, and after the lesson. In particular, we need to look deeper at how individual children respond to the lesson in ways that may be appropriate for their long-term development of powerful mathematical thinking.

In the pages which follow, I reproduce overheads from my presentation that look at each of the lessons to see where it fits in the overall plan of building ideas from a blend of embodiment and symbolism to build use and prove powerful mathematical concepts. This is, in no way, intended to be a once-and-for-all analysis. It is offered as a preliminary analysis for those developing lesson study to initiate discussion on how to implement the techniques of lesson study within a long-term framework that focuses on improving the learning of mathematics for each and every student.

A Long-Term Learning Framework for Lesson Study

Placing Plates (Grade 2)

December 2, 2006, University of Tsukuba Elementary School - Takao Seiyama



There are candies placed on small plates that are shaped like triangles and a quadrilateral, just like those shown below.

One of the tasks of this lesson is to make a large hexagonal plate by fitting together small plates like those shown above. Rules for making a large plate are as follows:

> You must fit together the small plates and make a shape that matches the large plate exactly.

Below are some examples. After you complete the task, count the number of candies.

① 20 candies $\left(\begin{array}{l} 5\times 2=10 \quad 2\times 2=4 \\ 10+6+4=20 \end{array}\right)$

② 18 candies $\left[\, 2\times 6=12 \quad 12+6=18 \,\right]$

Students will notice the difference between the number of candies on the various small plates by using multiplication which the students learned before to find out the number of candies. After students present various solutions to this problem, I would like to expand the lesson by paying attention to students' awareness of the problems involved.

Using ideas in a non-routine, *problem-solving* activity.

[**Proving** by physical embodied experiment]

Met-before: shapes, simple arithmetic

Activity: how to think flexibly in a specific problem situation

Long-term: flexible thinking with specified rules, encouraging a problem-solving attitude in an idiosyncratic problem.

A Long-Term Learning Framework for Lesson Study

Placing Plates (Grade 2)

December 2, 2006, University of Tsukuba Elementary School - Takao Seiyama

Experimenting with the problem

A Long-Term Learning Framework for Lesson Study

Placing Plates (Grade 2)

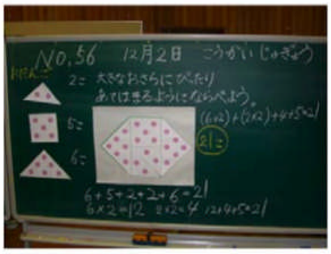
December 2, 2006, University of Tsukuba Elementary School - Takao Seiyama

Sharing data

A Long-Term Learning Framework for Lesson Study

Placing Plates (Grade 2)

December 2, 2006, University of Tsukuba Elementary School - Takao Seiyama



Organizing data

A Long-Term Learning Framework for Lesson Study

Placing Plates (Grade 2)

December 2, 2006, University of Tsukuba Elementary School - Takao Seiyama

An enjoyable well-planned activity allowing a wide range of levels of performance.

What is the contribution to future development?

Some practice of arithmetic
Flexible problem-solving
(e.g. finding all possible combinations)
Some idiosyncratic data eg squares can have 5 or 6 candies on them.
Questions:
What is the *important long-term role* of this lesson that the children should focus on?
What do individual children learn from this experience in the long-term?

A Long-Term Learning Framework for Lesson Study

Placing Plates (Grade 2)

December 2, 2006, University of Tsukuba Elementary School - Takao Seiyama

There are two objectives for this lesson. The first one is to foster students' geometric sense through composition of geometric shapes. And the second one is to foster students' ability to think logically and understand mathematical expressions by asking them to think about the composition of geometric shapes and their matching mathematical expressions.

3. Goals of the Unit

○ To foster student understanding of triangles and quadrilaterals through concrete manipulative activities.

○ To enrich the basic learning experiences of students by composing and drawing triangles and quadrilaterals.

4. Instructional Plan (Total: 6 periods)

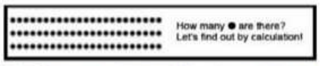
Phase 1: Meaning of triangles and quadrilaterals —— 2 periods
Phase 2: Composition and construction of triangles and quadrilaterals —— 2 periods
Phase 3: Summary and practice —— 1 period

Questions:
What is the *important long-term role* of this lesson that the children should focus on?
What do individual children learn from this experience in the long-term?

A Long-Term Learning Framework for Lesson Study

Multiplication Algorithm (Grade 3)

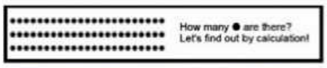
December 2, 2006, Sapporo City Maruyama Elementary School - Hideyuki Muramoto



A Long-Term Learning Framework for Lesson Study

Multiplication Algorithm (Grade 3)

December 2, 2006, Sapporo City Maruyama Elementary School - Hideyuki Muramoto



Building ideas in a flexible manner.

Met-before: single-digit multiplication subdividing a problem into smaller problems

Activity: constructing different ways of calculating 3 times 23

Long-term: flexible thinking about multiplication, revealing the standard algorithm as the most efficient.

A Long-Term Learning Framework for Lesson Study

Multiplication Algorithm (Grade 3)

December 2, 2006, Sapporo City Maruyama Elementary School - Hideyuki Muramoto

Experimenting with the problem

A Long-Term Learning Framework for Lesson Study

Multiplication Algorithm (Grade 3)

December 2, 2006, Sapporo City Maruyama Elementary School - Hideyuki Muramoto

Discussing ideas

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Multiplication Algorithm (Grade 3)

December 2, 2006, Sapporo City Maruyama Elementary School - Hideyuki Muramoto

Explaining to the teacher

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Multiplication Algorithm (Grade 3)

December 2, 2006, Sapporo City Maruyama Elementary School - Hideyuki Muramoto

Displaying different solutions

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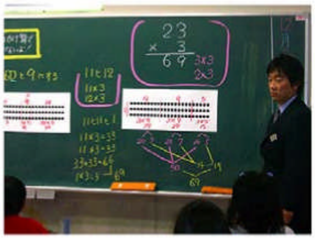
Multiplication Algorithm (Grade 3)

December 2, 2006, Sapporo City Maruyama Elementary School - Hideyuki Muramoto



Comparing solutions for efficiency

A Long-Term Learning Framework for Lesson Study

Multiplication Algorithm (Grade 3)

December 2, 2006, Sapporo City Maruyama Elementary School - Hideyuki Muramoto

A well-organised lesson in a sequence designed to give a flexible insight into multiplication.

What is the contribution to future development? Different children brought different met-befores. Some struggled with the arithmetic, some already knew the long-multiplication algorithm.

Questions:
What is the *important long-term role* of this lesson that the children should focus on?
What do individual children learn from this experience in the long-term?

A Long-Term Learning Framework for Lesson Study

Multiplication Algorithm (Grade 3)

December 2, 2006, Sapporo City Maruyama Elementary School - Hideyuki Muramoto

Goals of the Unit:

- Lessons that enable students to consciously think about the connection between what they learned before and what they are learning now
- Lessons in which students learn from each other and that help them consciously think about their own solution processes
- An evaluation method that helps foster students' logical thinking abilities.
- Unit plan
- This lesson (goals, process of lesson)

Questions:
What is the *important long-term role* of this lesson that the children should focus on?
What do individual children learn from this experience in the long-term?

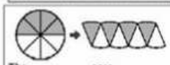
A Long-Term Learning Framework for Lesson Study

Area of a Circle (Grade 5)

December 5, 2006, University of Tsukuba Elementary School - Yasuhiro Hosomizu

1. Present the problem

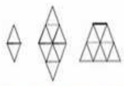
Come up with ways to find the area of the circle by using the sectors that are made by segmenting the circle into eight



The area of parallelogram=base×height

2. Think about different ways to rearrange the shape so that other formulas for finding the areas of basic shapes can be used

Rearrange the shape and find different formulas to find the area



dividing into 8 and 16 pieces

Building ideas using embodiment in a flexible manner.

Met-before:
area unchanged when parts are moved without overlap. Possible problem with curved edges.

Activity: making up areas from a subdivided circle

Long-term: meaningful understanding of the area of a circle.

A Long-Term Learning Framework for Lesson Study

Area of a Circle (Grade 5)

December 5, 2006, University of Tsukuba Elementary School - Yasuhiro Hosomizu

Thinking about the problem

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Area of a Circle (Grade 5)

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Making up solutions

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Explaining

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Summarizing

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Area of a Circle (Grade 5)
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A well-organised lesson in a sequence designed to give a flexible insight into various ways of seeing the area of a circle.

What is the contribution to future development? Designed to give meaning to the area of a circle. Questions remain about the curved edges in the area which could be seen in terms of 'local straightness' in calculus.

Questions:
What is the *important long-term role* of this lesson that the children should focus on?
What do individual children learn from this experience in the long-term?

A Long-Term Learning Framework for Lesson Study
Area of a Circle (Grade 5)
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3. Unit plan (Circle, total 10 lesson periods)
1st section Circle and regular polygons (2 lesson periods)
2nd section Length of circumference (3 lesson periods)
3rd section Area of circle (3 lesson periods) today's lesson 2/3
4th section Summary and application (2 lesson periods)
4. Goal of the lesson
1) Goal
Students will be able to come up with ways to find the area of a circle by rearranging the shape of the circle so that they can use previously learned formulas, and be able to derive the formula for finding the area of a circle.

Questions:
What is the *important long-term role* of this lesson that the children should focus on?
What do individual children learn from this experience in the long-term?

A Long-Term Learning Framework for Lesson Study
Thinking Systematically (Grade 6)
December 6, 2006, Sapporo City Hokuto Elementary School - Atsutomo Morii

4. Process of the lesson

Students' activities and thinking process	Teacher's support
We bought pencils and ballpoint pens and the total number of items were 10 and the price was 460 yen. The price of each pencil was 40 yen and the ballpoint pen was 70 yen. How many pencils and how many ballpoint pens did we buy? If we calculate it… / If we make a table…	○ Listening to the students' muttering (or voices) and pick up the idea to use a table to solve this problem. Then ask the students to fill in the table on the worksheet.

# of pencils	0	1	2	3	4	5	6	7	8	9	10
# of ballpoint pens	10	9	8	7	6	5	4	3	2	1	0
Total price (yen)	700	670	640	610	[illegible]	[illegible]	[illegible]	490	460	430	400

The table in the textbook shows the number of pencils and ballpoint pens from 1 to 9, but in this lesson I decided to use the number from 0 to 10. This is decision relates to my hope for a certain kind of mathematical thinking that I want my students to acquire.

More sophisticated solutions:

If we buy only ballpoint pens the total price would be 700 yen.
(700 – 460) ÷ 30 = 8
Thus, the answer is 8 pencils and 2 ballpoint pens.

If we buy only pencils the total price would be 400 yen.
(460 – 400) ÷ 30 = 2
Thus, the answer is 2 ballpoint pens and 8 pencils.

Building ideas relating 2 variables using tables.

Using *problem-solving* to use the data systematically

Met-before: Previous experience of relationships & tables.

Activity: more subtle solutions possible, but main focus on tables.

Long-term: to realise tables are systematic, but tedious, to generate the need for a more powerful way of expressing and solving the problem.

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Thinking Systematically (Grade 6)
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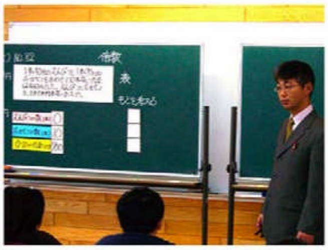

Starting a table with zero

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Building examples of data as columns

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Organising the complete table

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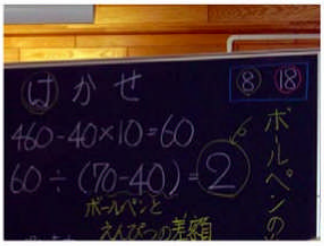

A more sophisticated solution

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Thinking Systematically (Grade 6)

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Intention of this lesson and mathematical thinking would like to foster among the students

During 4th grade, students learned how two quantities change in the unit called "investigating changes in quantities." In the 4th grade, they also learned how to express the relationship between two quantities using tables and math sentences. In addition, the students had experience reading the changes of the quantities and their characteristics.

In the 5th grade, based on their experience in 4th grade, students learned to solve problems by finding the relationship between two quantities and their regularity using tables.

The aim of this lesson is to use knowledge from prior grade levels to solve problems using tables that have more items. This lesson is included in the mathematics textbook. This lesson is not included as a part of a unit but it is set up as individual lesson. Title of the next unit is "proportional relationships." In the unit, students will construct tables, finding regularity, and expressing the relationship using math sentences. I believe this lesson is included here to help students prepare to learn about proportional relationships.

In this lesson, I anticipate that the students might solve this problem by coming up with an appropriate value and then calculating or by constructing a table. I believe that constructing a table is not a difficult task for the students because of their prior learning experiences. Moreover, I believe that many of the students will use a table to solve the problem.

The table in the textbook shows the number of pencils and ballpoint pens from 1 to 9, but in this lesson I decided to use the number from 0 to 10. This is decision relates to my hope for a certain kind of mathematical thinking that I want my students to acquire.

I would like to focus on a kind of mathematical thinking, i.e. hypothetical thinking.

Something like, "If it is ..... then ....."

By changing the quantities of the items in the problem on their own, the students can come up with better solution methods. In order to do that I think it is important for the students to see an extreme case in the table such as "I bought 10 items of one kind and 0 items of the other kind."

Questions:

What is the *important long-term role* of this lesson that the children should focus on?

What do individual children learn from this experience in the long-term?

A Long-Term Learning Framework for Lesson Study

Summary

Four lessons:

- 2 building new concepts: multiplication, area of circle,
- 2 using problem-solving: plates, thinking systematically.

All based on shared working, structured by the teacher; and all fitting into the long-term framework specified here.

The two building new concepts are part of a long-term development (as is thinking systematically).

The problem-solving activities have general principles of self-construction, sharing, making sense for one's self, etc, but do they have any problem-solving strategies?

There are general comments about desired learning.

What about analysis of *individual* development?

A Long-Term Learning Framework for Lesson Study

Comment

After formulating a theory of conceptual blending and possible misconceptions, this rarely features in the analysis!

In part the particular lessons do not reveal much about these issues.

Nor have we collected much *individual* data from the lessons.

The lessons are all planned for large classes with the focus on what is to be learned in a commendably flexible way, rather than on the range of possible individual learning that may reveal significant differences in performance.

In Britain, attention is turning to the needs of ‘pupils at risk’ who need extra support and to the ‘gifted and talented’ who need extra challenges.

> É for pupils at risk of falling behind, early intervention and special support to help them catch up. This is already underway with the ‘Every Child a Reader’ programme for literacy, which is now being matched with the ‘Every Child Counts’ initiative for numeracy, alongside one-to-one tuition for up to another 600,000 children.Gordon Brown, The Guardian, May 15, 2007

However, it is not a linear race, with some ‘falling behind’ and others ‘racing ahead’. It is also a question of different kinds of learning and different ways of coping.

Assuming our major purpose is to improve the long-term learning of mathematics for each and every one of our children, I suggest that there is a need for lesson study to be placed in a long-term framework to design and monitor the long-term development of individuals, to gain insight not only what needs to be learnt and how, but also why some develop flexible, powerful mathematical thinking and others have serious difficulty.

The framework offered is based on the different styles of cognitive growth in embodiment and symbolism over the long -term, and the way in which different individuals build on mental structures based on ideas met-before.

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REASONING ABOUT THREE-EIGHTHS: FROM PARTITIONED FRACTIONS TOWARDS QUANTITY FRACTIONS

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Curriculum documents describe the importance of questioning, reasoning and reflecting as contributing to Working Mathematically. A research lesson on the development of units of different sizes (eighths) associated with measurement and fractions, is developed as a vehicle for developing mathematical reasoning through argumentation in a composite Year 3–4 class. Making a transition from embodied fractions (parts of a whole) to recognising the equal whole needed for comparison of fractions as mathematical objects extends the current Mathematics curriculum in New South Wales. The lesson study also highlighted the need to develop taken as shared meaningfor fraction units used in classroom argumentation.

REPRESENTATIONAL CONTEXTS

Working Mathematically draws on ways on ways of seeing, questioning, interpreting, reasoning and communicating. This type of mathematical thinking was summarised by Schoenfeld as follows:

> Learning to think mathematically means (a) developing a mathematical point of view— valuing the process of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade, and using those tools in the service of understanding structure—mathematical sense-making. (Schoenfeld, 1989, as cited in Ball, 1993, p. 157)

The tools used in the service of understanding structure are often derived from *models of* the mathematics. For example, partitioned circles or rectangles are used as regional models of partitioned fractions (Watanabe, 2002) which can contribute to associated concept images of fractions . Yet physical models have limitations in that they at best imply the mathematical concept, and can add unwarranted components to the intended concept image. In particular, students' evoked fraction concepts suggest that equality of area is not always the feature abstracted from regional models used in teaching fractions (Gould, 2005). Further, to be able to interpret the part -whole comparison of area intended by the regional model, children need to be familiar with the context, which in this example includes the concept of area (Lampert, 1989). Choosing an appropriate representational domain is an important teaching decision in developing students' mathematical understanding.

In practice, fractions exist in essentially two forms: embodied representations of comparisons, sometimes called partitioned fractions, and mathematical objects, also known as quantity fractions. A *partitioned fraction* (Yoshida, 2004) can be described as the fraction formed when partitioning objects into *b* equal parts and selecting *a* out of *b* parts to arrive at the partitioned fraction *a/b*. A partitioned fraction can be of either discrete or continuous objects but a partitioned fraction is always a fraction of something. By comparison, quantity fractions are mathematical objects defined as fractions that refer to a universal unit. Asking the question, which is larger, one-half or three-eighths, only makes sense if the question is one of quantity fractions. Quantity fractions implicitly reference a universal unit, a unique unit-whole, which is independent of any situation. If one-half and three-eighths as mathematical objects do not refer to a universal whole, we cannot compare them.

The learning of fractions is subject to a paradox that is central to mathematical thinking (Lehrer & Lesh, 2003). On one hand, a fraction such as 2/3 takes its meaning from the situations to which it refers (partitioned fraction); on the other hand, it derives its mathematical power by divorcing itself from those situations (quantity fraction). Working with partitioned embodiments of the fraction “3/7” can elicit a parts-of-a-whole meaning as “three out of seven”, but without divorcing the fraction notation from this context interpreting “7/3” does not make sense. It is difficult for students to become aware of a unit-whole when the unit-whole is often implicit in everyday situations involving fractions. To make the transition from partitioned fractions to quantity fractions, students need to develop a sense of the size of fractions. A sense of the size of fractions is what Saenz -Ludlow (1994) refers to as conceptualising fractions as quantities. Mathematical thinking associated with working with units of various types and in particular, the introduction of abstract units, are central to mathematics.

PLANNING THE LESSON

In the dominant instructional model used to teach fractions in New South Wales, children learn to divide objects into equal parts. Next, children learn to count the number of parts of interest and place the result of this count above the count of the total number of parts using the standard fraction notation. This part-whole recording method is used to introduce fraction notation. The transition from counting parts of a model to recording fraction notation is followed by instruction on the traditional algorithmic manipulation of the whole number components of the fraction notation, known as operating with fractions.

Unfortunately, the predominant instructional model is not successful for many students and fractions are a particularly troublesome area of the elementary mathematics curriculum in NSW. The language associated with fractions in English contributes to a number of misconceptions for students. Unlike most Asian languages, English uses the same terms for naming ordinals and fractions (e.g. third, sixth, ninth). It is also easy for students to not hear the soft sounds at the end of fraction names, which can lead to confusion between whole numbers (e.g. six) and fractions (e.g. sixth). Thus, although six sixes are thirty -six, six sixths are one.

Following the mathematics syllabus, children describe halves in everyday contexts in their first year of school, Kindergarten. Ironically, everyday representational contexts for halves include examples such as cutting a piece of fruit into halves, where the means of determining the equality of the pieces relies on an understanding of volume. In the following two years (Years 1 and 2) children are expected to model and describe a half or a quarter of a whole object or collection of objects as well as to use the fraction notation 1/2 and 1/4. In Years 3 and 4 children model, compare and represent fractions with denominators 2, 4 and 8 as well as find equivalence between halves, quarters and eighths.

For fractions to make the transition from embodied partitions to mathematical objects the idea of a universal unit-whole needs to be established. This universal unit-whole is a 'one' that remains the same size in all contexts and is similar to a standard unit of measure. Making a transition from embodied fractions (parts of a whole) to recognising the equal whole needed for comparison of fractions as mathematical objects is the unit goal for lesson study outlined below. The idea of lesson study was new to the teacher and the school. Further, the idea of the need to identify the equal whole and the specific role of representational contexts are not part of current teaching practice in elementary schools in New South Wales. Consequently, the planned abstraction referred to in the unit goal is a very ambitious goal for the composite Year 3 and Year 4 class taking part in the lesson study.

Developing thinking through argumentation

The key window for considering mathematical thinking in this lesson study is through justification in reasoned argument. Learning to argue about mathematical ideas is fundamental to understanding mathematics. Palincsar and Brown (1984) wrote that " ...understanding is more likely to occur when a child is required to explain, elaborate, or defend his position to others; the burden of explanation is often the push needed to make him or her evaluate, integrate and elaborate knowledge in new ways." Argument here is taken to mean a discursive exchange among participants for the purpose of convincing others through the use of mathematical modes of thought.

The ways in which students seek to justify claims, convince their classmates and teacher, and participate in the collective development of publicly accepted mathematical knowledge contribute to mathematical argument. In a culture that expects student understanding, teaching mathematics is more than merely telling or showing students; teachers must enable students to create meanings through their own thinking and reasoning. Classroom argumentation needs opportunities to move from authority-based arguments (because the teacher says so or the text states this) to reasoning with mathematical backing. That is, "how do you know?" is the key question. The expectation is that students arrive at consensus through reasoned argument, reconciling different approaches through demonstration using a common model.

THE LESSON: THREE-EIGHTHS OF THE BOARD

The link between the process of division and the creation of fractions is not always clear to students. To simplify the creation of this link, the attribute of length is used instead of area to create partitioned fractions. Although regional models are often used to introduce fractions, some students focus on the number of regions and not the area of the regions compared to the whole shape in abstracting the fraction relationship.

By the start of Year 3, children can model and describe a half or a quarter of a whole object or collection of objects as well as being familiar with the fraction notation 1/2 and 1/4 (syllabus reference NS 1.4). Multigrade classes are quite common in New South Wales and the class taking part in the lesson study described here had 6 Year 3 students and 21 Year 4 students. The Mathematics K–6 syllabus describes content in stages corresponding to two school years with the exception of the Kindergarten year, referred to as Early Stage 1. As the students in the study had covered the fractions content from Stage 1 (Years 1 and 2) this lesson was designed as an introduction to eighths and the relationship between eighths, quarters and half (Appendix A).

The lesson started by inviting three students to estimate where three-eighths of the width of the class white-board would be, mark the point and put their initials next to their estimates. The students were chosen by the teacher based on her knowledge of the students to obtain variation in the estimates. The fraction 3/8 was used with the attribute of length to focus on composition of partitioning through repeated halving. As the students had previously covered work on 1/2 and ¼ the use of 3/8 also provided scope for iterating the unit fraction. Having the students record their initials next to the estimates gains ownership of the estimate by the student and encourages a desire to find out. It also makes discussion about the estimates easier by providing a name for the estimate.

A piece of string the same length as the white-board was cut and used by students to form one-eighth and in justifying which estimate was closest. The class discussion also provided opportunities to relate the values of half, quarters and eighths of the same length. The students returned to their desks following the discussion and located positions corresponding to various numbers of eighths of different sized intervals (see Appendix B). After discussing the location of three-eighths on different sized intervals the final question was posed: *Could 4/8 ever be less than 2/8?*

Students' explanations

Having established the unit of one -eighth, the teacher used one -half of the length of the whiteboard as a benchmark to determine how close the estimates were to three-eighths. The teacher asks how close was Jack's estimate to three-eighths and a very rapid exchange takes place between the students. One student (Charlotte) says that Jack was not as close as Emily, a mark greater than one-half of the length of the board. Another student thinks about the response and quickly says, "That's a half". The teacher picks up on the exchange.

Teacher: *That's an interesting comment. Charlotte's comment was he wasn't as close as Emily.*

Teacher: *Remember we were wanting to find three-eighths of the length of the white-board. If this is halfway, how much of the white-board do you think Emily found?*

Stephen: *Two, three quarters, two and a half quarters.*

Teacher: *Two and a half quarters? OK. Amy, how...*

Amy: *Four and a half eighths.*

Teacher: *Jessica.*

Jessica: *It could be five-eighths.*

Jessica then went on to describe the size of the unit (partitioned) fraction one-eighth and that one half of the board corresponds to four-eighths (Fig. 1), so adding on one eighth more making five-eighths would be a position very close to Emily's mark.

Figure 1. Describing one-half as four-eighths

The discussion arising from using the benchmark of one-half to determine which of the estimates was closest to three-eighths, and the subsequent description of each estimate location in terms of eighths, was very informative. The orchestration of the discussion did enable the teacher to intervene to seek justifications for the different beliefs as to which estimate was closest to three-eighths.

However, developing *taken as shared* meaning for fractions as quantities is not simple. In a later part of the lesson, when one student explains why three-eighths is at a different location on the first two intervals, he suggests that it might be because "all of us used different strategies to work out the answer". This suggested was followed by a surprising exchange.

Teacher: *What strategies did you use?*

Student: *I cutted them up into all kinds of different quarters and then I went to... uhm, and then I counted it from one for every quarter.*

Teacher: *Why did you cut it up into quart ers?*

Student: *Because ... uhm...* [Pause]É *I cut it up into different quarters because then I'd know how much of each part of it is the same.*

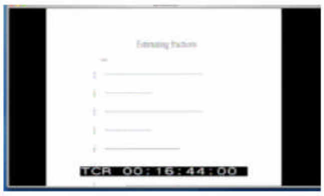
The exchange is puzzling for both the student and the teacher, as they appear to have different meanings for quarters. When students are challenged to explain their reasoning the evoked concept images (Tall & Vinner, 1981) can reveal unexpected interpretations of fractions such as quarters. The evoked image related to the term quarter was quite different between student and teacher. The student was using the term quarter for a general fraction part. Rather than meaning one of four equal parts comprising the whole, the term quarter meant ‘equal piece’ for this student. This interpretation was confirmed when the video of the lesson was replayed to the student. Just as the common use of fraction terms can lead to exchanges that suggest that one-half can be bigger than another half (*You take the bigger half)* the same appears to occur with the term quarter. As the teacher and the student did not have a *taken as shared meaning* for a ‘quarter’ the discussion did not advance the understanding of fraction units. Recognising that the nature of mathematical argument may vary between cultures (Sekiguchi & Miyazaki, 2000) confronting the individual’s misconception about fraction units was not always straightforward. The teacher can only respect a student’s idea if the teacher and the other students can understand the idea. Although the teacher encourages other students to think about how quarters might be of use in determining three-eighths, the problem cannot be truly shared if individuals hold different concept images for quarters.

REFINING THE LESSON

Although the lesson was generally quite effective at encouraging students’ justification of fraction units and in preparing to make a transition from embodied fractions (parts of a whole) to recognising the equal whole needed for comparison of fractions as mathematical objects, revisions are needed to better address two areas. The first area relates to the number of students who reached the desired goal of the lesson. About one-quarter of the class had difficulty generalising the process of repeated halving to create eighths of different sized units. This suggests that it might be helpful to carry out the process of finding three-eighths of a different length, estimating and then using repeated halving, with students clarifying the process before moving to the worksheet. The second area, which is integrally related, is developing *taken as shared meanings* for one-quarter and one-half. Discussion of the strategies used to partition intervals into eighths relied on shared meanings for quarters that did not appear to exist within the class. The next lesson in this unit looked at students partitioning pieces of string using small pegs to construct related fractions. This activity (related to eighths, quarters and halves) may have helped in creating the *taken as shared meanings* for these fractions before discussion arising from the worksheet activity (Appendix B).

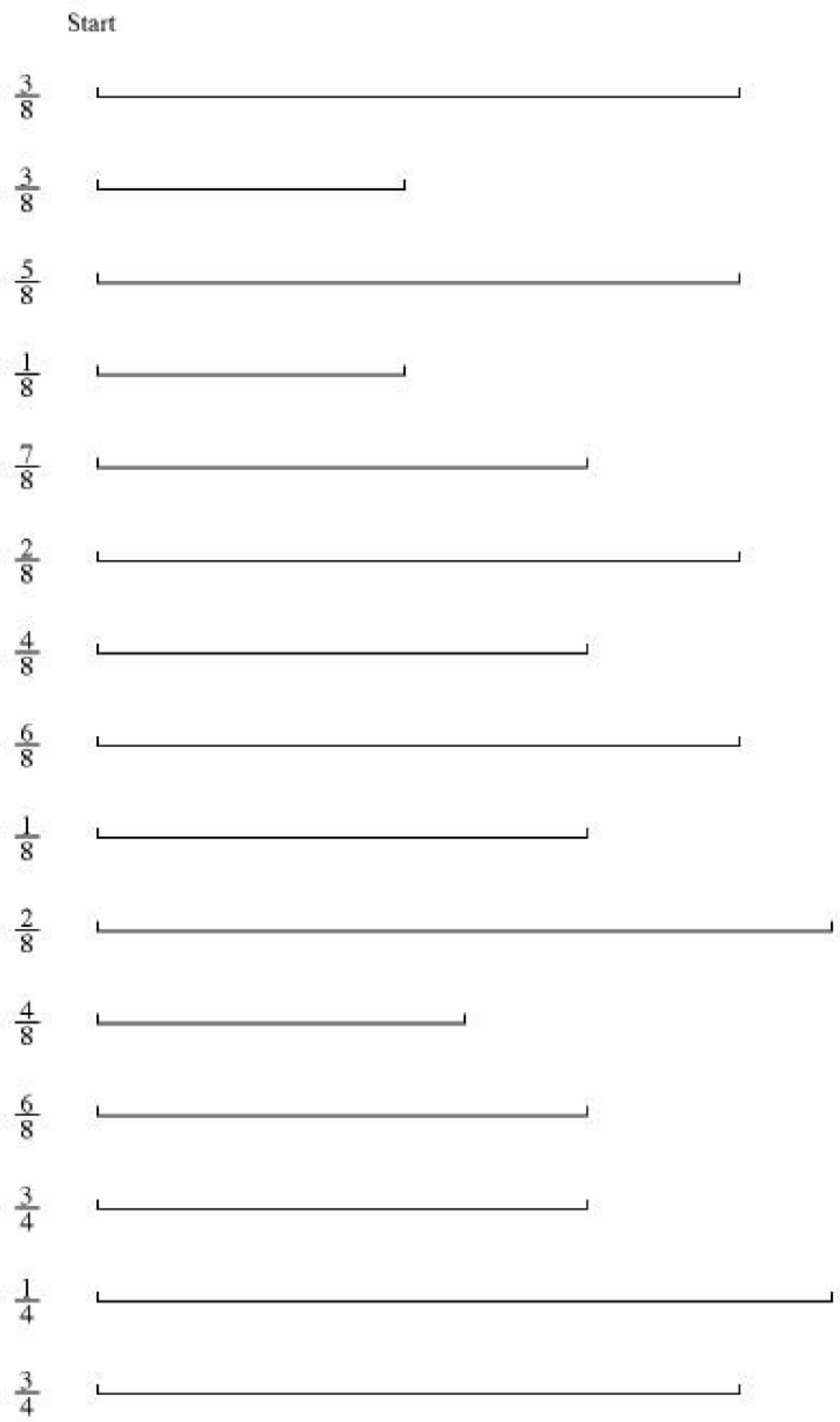
APPENDIX A: ESTIMATING 3/8 OF THE WIDTH OF THE BOARD (GRADE 3-4)

Goal: The fraction 3/8 is used with the attribute of length to focus on composition of partitioning through repeated halving, to link to students' concept images of one-half and one-quarter.

	OUTLINE	COMMENTS
Problem posing	Challenge the students to estimate 3/8 of the distance from the left side of the board. Invite three students to mark the distance and to add their initials to the marks they make. T: *How could we determine which mark is closest?* S1: *Measure the distance, divide by eight and multiply by three.* S2: *Find half way, then find half of that to get a quarter, then find half of the distance between one-half and one-quarter.* S3: *Get a piece of string the length of the board and then fold it into eight.*	If the students have difficulty with 3/8 change the question to estimating 1/8 first. Repeated halving is the method used by most students. 
Justifying solution methods	Discuss the strategies. Using a piece of string, invite a student to demonstrate how to check which estimate is closest and justify his or her reasoning. Compare the location of 3/8 and half way. Determine the location of the three crosses in terms of eighths. Use a worksheet involving a number of intervals of different lengths that require students to locate fractions relating to different numbers of eighths (e.g. 3/8, 5/8, 7/8, 1/8) NS2.4 The first two questions ask students to estimate the location of three-eighths on different sized intervals. Check to see if students are locating three-eighths at the same distance from the left, and discuss. *Should the answers be in the same place?*	 Check understanding of the number of folds and the number of parts. 
Comparing units	Put the following question on the board. *Could 4/8 ever be less than 2/8?* Use think-pair-share or a quick feedback back method (e.g. thumbs up, down, sideways) to determine student responses.	This question is designed to prompt the introduction of the need for the equal whole in the transition from partitioned fractions (fractions of things) to quantity fractions (fractions as numbers).

Appendix B

Estimating fractions



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INCORPORATING MATHEMATICAL THINKING IN ADDITION AND SUBTRACTION OF FRACTION: REAL ISSUES AND CHALLENGES

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Since the introduction of the new primary mathematics curriculum in Brunei starting January2006, more teachers of elementary schools are seeking a suitable way of incorporating mathematical thinking in each mathematics lesson that they teach. For this particular purpose, teachers were introduced to lesson study and it is hoped that with the guidance and support given by the team involved, teachers would build their confidence and make mathematical thinking a regular feature of their lessons. There are still many problems that the teachers face such as students not used to explaining their thoughts in class and some insisting on using certain procedures that they had learned before, without being able to explain how and why the procedure works. This paper will relate a classroom case and look at the real issues and challenges that Bruneian teachers faced in incorporating mathematical thinking when teaching of the topic on addition and subtraction of fractions.

Introduction

Mathematical thinking is the mathematical mode of thought that we use to solve any problem in our daily life including at schools (Khalid, 2006). It can be defined as applying mathematical techniques, concepts and processes, either explicitly or implicitly, in the solution of problems (Khalid, 2006). It is, according to Katagiri (2006), the most important ability that mathematics courses need to instill because it makes students able to think and make independent judgement. He also said that mathematical thinking allows for an understanding of the necessity of using knowledge and skills as well as learning how to learn by oneself, and the attainment of the abilities required for independent learning. Stacey (2006) reiterated this fact by saying that mathematical thinking is important because it is an important goal of schooling; it is important for teaching mathematics; and it is an important way of learning mathematics. In fact, the framework used by PISA shows that mathematical literacy involves many components of mathematical thinking, including reasoning, modelling and making connections between ideas. It is therefore imperative that mathematical thinking be stressed in any school curriculum and this is reflected in the new curriculum for primary mathematics of Brunei Darussalam, which was put into implementation from early 2006 (Khalid, 2006).

Children are encouraged to use thinking skills and problem solving strategies during mathematics lessons and not just learn mathematical skills and concepts from listening to the teachers. It is feared that if mathematical thinking is not emphasized, our children would end up learning mathematics by rote memorization, without understanding and without the ability to think intelligently.

More on Mathematical Thinking

The new Bruneian "Mathematics Syllabus for Lower and Upper Primary Schools" (CDD, 2006a; 2006b) considers mathematical thinking as among the processes, skills and values that need to be developed through the teaching and learning of mathematical content. This bears similarity to how Professor Katagiri (2006) defined mathematical thinking. According to him, mathematical thinking can be divided into three categories:

I. Mathematical Attitudes
II. Mathematical Thinking Related to Mathematical Methods
III. Mathematical Thinking Related to Mathematical Contents

The first category is considered as the driving force behind the two latter categories.

"Mathematical attitudes" is a very important affective factor in determining students' behavior in mathematical thinking and problem solving because students' attempts in mathematical thinking depend on how interested they are in problem solving or the lesson. Students' expectation that mathematics will be useful (which involve beliefs) and their personal attributes such as confidence, persistence and organization are mentioned by Stacey (2006) as some of the skills and abilities required for problem solving. Attitudes and values are also mentioned in the Brunei curriculum document (CDD, 2006a; 2006b).

"Mathematical thinking related to mathematical methods" was listed in detail by Katagiri (2006) as consisting of inductive thinking, analogical thinking, deductive thinking, integrative thinking, developmental thinking, abstract thinking, thinking that simplifies, thinking that generalizes, thinking that specializes, thinking that symbolizes and thinking expressed with numbers, quantifiers and figures. Stacey, (2006) quoting from Mason, Burton and Stacey (1982) defined this category as mathematical process that is made up of:

- specializing – trying special cases, looking at examples
- generalizing – looking for patterns and relationships
- conjecturing – predicting relationships and results
- convincing – finding and communicating reasons why something is true

The resemblance of this category in the Brunei curriculum document (CDD, 2006a, 2006b) would be the processes which include mathematical thinking and communication.

"Mathematical thinking related to mathematical content" include ideas of sets, units, expressions, operations, algorithms, approximation, fundamental properties and formulas. These can be compared to mathematical skills (as well as estimation and mental computation) in the Brunei curriculum or deep mathematical knowledge as stated in Stacey's requirement for problem solving.

In the Bruneian syllabus, mathematical thinking and problem solving are mentioned together. Teachers must encourage children to use thinking skills and problem solving strategies during mathematics lessons (CDD, 2006b, p. 7). Among the sub-processes of the mathematical thinking and problem solving processes that are listed in the syllabus are: guessing and checking, drawing diagrams, making lists, looking for patterns, working backwards, classifying, identifying attributes, sequencing, generalising, verifying, visualising, substituting, re-arranging, putting observation into words, making predictions as well as simplifying the problem and solving part of problems.

The curriculum recommends the use of a variety of representations to facilitate the development of the content knowledge and processes. Active learning is encouraged and the use of different representations is to be implemented according to the age and stages of the pupils. In the early years, concrete materials are supposed to help children develop basic mathematical concepts. As children move on, diagrams, real-world examples, verbal representations, ICT and symbolic representation will help children proceed from the concrete to more abstract ways of thinking. The use of symbolism to shift from process to concept is what Tall (2006) termed as 'procept'.

The Lesson Study Group

Lesson study was recommended by Khalid (2006) as a professional development for teachers to familiarize themselves with incorporating mathematical thinking in their lessons in Brunei. Since many Bruneiean teachers are not very familiar with lesson study, a long-term strategy in the form of research project was developed to introduce lesson study, in order to make it a regular feature in teachers' professional development and training. A team was established to ensure the smooth running of the project. The team comprises of me (as the project leader), Md. Khairul Amilin Hj Tengah as well as Dr. Hjh. Zaitun Hj. Taha from Universiti Brunei Darussalam, Mr. Palanisamy Veloo from the Curriculum Development Department, and Mr Masaki Takahashi from Sultan Saiful Rijal Technical College. We managed to secure a research grant from the university to help us finance this research study which was approved in June. We managed to attract four teachers attached at a particular Secondary School to join our project and we started the training way before the research grant was approved to make sure they have enough time to consolidate the ideas behind lesson study before we start with actual classes. During training, the philosophy and process of lesson study were explained and with the help of the lesson study videos that were received from Japan, the teachers could clearly see what was meant.

We started our actual lesson around mid April and since the research grant was not yet approved at that time, we manage to record the lessons with a DVD camera belonging to one of us, without even a tripod. When we were about to start the actual teaching process, two teachers withdrew. Of the remaining two, only one attended the meeting regularly to discuss her lesson plans with us. We managed to record the lessons of both teachers teaching different topics. However, for the purpose of this presentation, we will only look at the lessons of one particular teacher.

The Study

The study involved two secondary one (about grade/year 7) classes, 1A and 1E. The two classes comprised 34 and 38 pupils respectively. The main aim of the lesson is to incorporate mathematical thinking into teaching and learning of addition and subtraction of fractions. There are other aims stated in the teacher's lesson plan and they are as follows:

At the end of the lesson, students should be able to:

I. Perform the addition and subtraction of fraction involving:
   - Fractions with like denominators
   - Fractions with unlike denominators
   - Improper fractions and mixed number

II. Solve problems related to addition and subtraction of fraction

Lesson Plan Development

The lesson plan for the purpose of lesson study was written by the teacher after discussion with the team. She was told that since the purpose of the lesson is to incorporate mathematical thinking, she also needs to think of the kinds of questions to ask the student and what to expect from students' responses. She was determined to make students participate during the lesson because communication is necessary for developing mathematical reasoning. The lesson plan can be seen in Appendix A.

For introduction, the problem that she posed for the students was composed to make students interested in the lesson and to stimulate their thinking. This is considered an important part of the lesson because it dealt with mathematical attitude. It is one of the ways to motivate students, as was mentioned in Keller's (1983) ARCS model where students' attention is gained and maintained by innovative posing of problem. The teacher adapted the names of famous international stars and weird long names for this purpose. The teacher also tried to cover the syllabus to include like, unlike and improper fractions in her introductory problem.

Next, the teacher prepared teaching aids in order to help students visualize the problems in a concrete way which can be classified as thinking expressed with figures or manipulatives. Her paper folding activity is also an attempt to make students translate thinking that symbolizes to thinking expressed with figures. The ability of students to translate parts of a round pizza to rectangular parts also involved mathematical thinking because it requires the ability of students to simplify and translate the problem to another equivalent form. She also tried to make children think by generalizing when she asked students to look at patterns (or what happen) when the questions were changed to fractions with larger denominator. At the same time, she planned for students to conjecture about the result when this happens. The learning processes in conjecturing include defining, exploring and constructing premise/conclusion according to Fou-Lai Lin (2006) or predicting relationships and results according to Stacey (2006).

To further assess students' understanding of the lesson, the teacher prepared different sets problems involving fraction magic-squares. Number puzzles and tricks are excellent for featuring mathematical thinking prominently in lesson (Stacey, 2006). Students were made to work in pairs and each pair was given a different set of magic-square. Working in pairs requires ability such as communication and interpersonal skills. The lesson was planned well and the team was eager to observe the lesson.

Lesson Observation and Comments

Below are the comments on the implementation of the lesson based on the observation data that was collected. I will try to identify the elements of mathematical thinking that are present during the lesson.

The teacher attempted to present the lesson via problem-solving strategy. When the problem was posed in the context described, the element of mathematical attitude (willingness to attempt and attempting to discover mathematical problems in daily life) was present. Students were observed to be very excited and can be heard repeating some of the weird and famous names. The teacher was therefore successful in making students interested in the lesson as well as stimulated their thinking. Later, she pasted the teaching aids in the form of paper pizzas on the white board. Interest, enthusiasm and attitude are important to arouse curiosity and she has done that successfully. She then proceeded by asking the students to give her the mathematical statements of the problems. Students had done fractions before and they can therefore build on experiences that were *met-before* (Tall, 2006). The children responded by giving chorus answers. Here, ideas are compressed into thinkable concepts using language and symbolism (Tall, 2006). At this stage, the teacher had used four of the representations suggested by the curriculum – real-life, diagram, verbal and symbolic.

She then proceeded to use the concrete representation (manipulatives), the paper-folding activity. Therefore the only representation that she did not use was the ICT, which I think would be one of the best representations for better understanding of addition and subtraction of fraction. She could have brought the students to the audiovisual room for ICT representation since the facilities there would allow the use of ICT. However, the use of concrete representation involved some amount of mathematical thinking when students need to transfer other representations to this representation. Since the shape of the paper used is not the same as the one in the diagram, children need idea of units (mathematical content) and focus on the constituent elements and their size and relationship (Katagiri, 2006). Children were encouraged to fold the rectangular papers guided by their teacher to understand how the answers were obtained. In the case of unlike fractions, most students prefer the use of lowest common multiple (LCM), which is the procedural knowledge that they had acquired before. Students do not really understand the idea of LCM since even for simple LCM of 4 and 8, they needed to use the algorithm that they learned before, and did not use logical thinking at all. The teacher however, did not take this opportunity to explain LCM in terms of finding equivalent fractions. This is a common feature of a mathematics lesson in Brunei, where teachers teach procedures and algorithm without explanation and students learnt them without understanding. So they were able to do mathematics without understanding what the lesson is all about. There are other instances where rules were given and children try to remember them by heart. When they are many rules to remember, they would be confused and make mistakes.

Students were able to generalize and conjecture (mathematical method), when the teacher guide them to look at the patterns and when fractions with large denominators were used. Since folding paper would be out of the question, this is considered as an important part of the lesson. However, it would have been better if she asked one of the children to communicate their reasoning by asking appropriate questions instead of just getting chorus answers. Most of the explanations given by students were not very convincing.

Before the end of the lesson, children were provided with magic-squares where they have to fill-in the empty boxes. Children were observed to communicate with each other and the teacher to discuss the problem. Although clear instructions were given to the students, almost all of them could not complete the problem.

During discussion with the team, the teacher was advised to teach addition and subtraction of fractions separately, because she had to rush through the lesson due to limited time. The teacher responded that her lesson was planned that way because the students had learned about addition and subtraction of fractions before. However, she will separate them into two lessons for the other class that she would teach that week since it seemed that students still need time to understand about fractions. Our comments were taken positively and we could see some improvements when she taught subtraction of fraction to class 1E. Here we could see that by teaching addition and subtraction separately, the phase of the lesson in 1E was just right. Furthermore, these students are supposed to be a weaker group than the previous one. We were not able to observe the teaching of addition of fraction for class 1E because many of the team members had other commitments on that day.

Discussion

So, what are the real issues and challenges related to the incorporation of mathematical thinking in each lesson in Brunei Darussalam? In my last paper (Khalid, 2006), I have mentioned three perceived issues and challenges:

1. The over-emphasis on examination and examination results.
2. Teachers readiness to teach students to think mathematically
3. Changing the expectations of the stake-holders.

I will however add another one and that is lack of students' participation during lesson.

The first issue is still the biggest issue to be addressed. The children involved in this study were children who sat for their "Primary Certificate Examination" the year before. Considering they have gone through fraction with their teachers during primary levels, one would expect a solid understanding of the topic from them. However, looking at the way they tried to solve the magic-square, it seemed like they were learning the topic the first time. Their persistence on using LCM to solve addition and subtraction of unlike fractions proved the case that they remember algorithm and do not actually understand what LCM stands for. This agrees with the findings of Lim (2000) and Clements (2002) that said that Bruneian children have low level of conceptual understanding of mathematical ideas involved and rely on procedural approaches or rote memorization. Tall (2006) reiterated the fact that procedures that are not compressed into thinkable concepts may give short-term success in passing tests, but if those procedures are not given a suitable meaning as thinkable concepts (in this case, procepts), then they may make future learning increasingly difficult.

The need to reform assessment and evaluation of school children becomes more crucial. Assessment should vary and should not solely depend on one sit-down examination to determine students' progression. At the elementary level, more performance-based and authentic assessment should be introduced. Traditional testing methods in mathematics have often provided limited measures of student learning, and equally importantly, have proved to be of limited value for guiding student learning. These methods are often inconsistent with the increasing emphasis being placed on the ability of students to think analytically, to understand and communicate, or to connect different aspects of knowledge in mathematics. I am however pleased to hear that the ministry of education is doing something about this. As a first step, they could at least set the examination questions into those that need mathematical thinking to solve.

The second issue concerning teachers' readiness to teach students to think mathematically is really an important issue. Teachers are not used to teach this way and have not been exposed to this kind of teaching. Although the idea and method was taught to them during teacher training, they tend to go back to the traditional method once they were posted to schools. I guess old habits die hard. In the attempt to expose teachers to mathematical thinking through lesson study, I have not had the opportunity to extend this to other schools, due to lack of time and resources since the grant for the purpose of lesson study was only approved in June. We also need committed teachers who are interested in lesson study to make it a success. We went through a setback when teachers decided to withdraw from the study. I guess the school should introduce a reward system for teachers who participate in lesson study or other professional development courses. However, I am still hopeful that it will succeed when I hear about the success of lesson study in other countries.

I consider the third and last issue as the most difficult to change. Stake-holders like parents and children should be made to realize that putting too much emphasis on examination results will in the long run lead to the children being disadvantaged. There should be an awareness that the nature of work has changed. Employers nowadays seek workers who are team-players, thinkers and problem solvers. Mathematical thinking provides a solid foundation to produce good problem-solvers. School administrators should not put too much pressure on the teachers to produce good result, because there is a tendency for teachers to take a short-cut to achieve this. Teachers become pre-occupied with preparing students for examinations (Majeed, Aldridge& Fraser, 2001) and students would come to regard mathematical "understanding" being the same as being able to answer examinations questions correctly (Clements, 2002). If we do not change, then the level of scholastic ability of the students will be among the top of the list produced by Katagiri (2006) in the hierarchy of scholastic abilities and mathematical thinking (from lower to higher) reproduced below:

1. The ability to memorize methods of formal calculation and to carry out these calculation
2. The ability to understand the rules of calculation and how to carry out formal calculation
3. The ability to understand the meaning of each operation, to decide which operations to use based on this understanding, and to solve simple problems
4. The ability to form problems by changing conditions or abstracting situations
5. The ability to creatively make problems and solve them

The higher the level, the more important it is to cultivate independent thinking in individuals. To this end, mathematical thinking is becoming even more and more necessary.

Students' reluctance be active participants in class can be corrected through constant encouragement from teachers. There are many reasons for them to be quiet in class, such as being afraid of making mistakes and not being fluent enough in English. However, classroom culture can change if teachers insist and encourage certain behaviors and I am glad to see that more and more of our teachers encouraging students to speak and not being afraid of making mistakes.

Conclusion

Mathematical thinking has been proven to be important and is therefore emphasized in the new Bruneian "Mathematics Syllabus for Lower and Upper Primary Schools". The success for implementing mathematical thinking needs concerted effort from everyone involved. The country needs a well educated workforce with the ability to think and analyze, using varied reasoning and problem-solving skills in an integrated manner for national development. In order to be able to independently solve problems and expand upon problems and solving methods, the ability to use **"**mathematical thinking**"** is considered even more important than knowledge and skill, because it enables to drive the necessary knowledge and skill (Katagiri, 2006).

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APPENDIX A

Lesson Plan of the First Lesson Observed

Name of Teacher: Hajah Nuzailah
Class: 1A (Year/Grade 7)
Topic: Addition and Subtraction of Fraction.

Learning Objective: So that children are able to think mathematically how to solve problems related to addition and subtraction of fractions (like and unlike fractions).

Other Objectives: 1) For students to be interested in the lesson (have the right attitude to involve in mathematical thinking)
2) For students to think mathematically in integrating what they know and relate it with what they are currently learning and to make sense of the lesson using the manipulatives provided.

Introduction: Pizza problem

These students ordered some pizzas from the pizza vendor and ate them according to the following proportion:

Name	Amount of pizza
1. John Beckham	1/3
2. Ramziah Khuzaifah	1/3
3. Shawn Wayne	1/5
4. Agus Muslimah Qawiyah	3/8
5. Hendrick Schumacher	1½

Questions Posed	**Corresponding thinking**
1. What is the amount of pizza John and Ramziah have together?	Students are supposed to add like fractions
2. If John and Ramziah gave up 1/3 of their total share, what is the amount they have left?	Students are supposed to get the sum from question1 and subtract ¼ from it. Subtract like fractions
3. What is the amount that both Shawn and Agus have together?	Adding unlike fraction
4. What is the amount if Shawn and Agus give up ¼ of their share to a friend?	Getting the sum from question 3 and subtract ¼ from it
5. What is the amount that Shawn and Hendrick have together?	Adding unlike fractions involving improper fraction
6. Shawn and Hendrick gave up 2/3 of their share. How much do they have left?	Getting the sum from Question 5 and subtract 2/3 from it
7. How many pizzas did they order altogether?	Getting the sum of all the fraction of pizzas And rounding them

Activity: Paper Folding

Students are given blank papers, and are encouraged to translate the questions using he rectangular paper to get answers (by folding and shading). Students were asked to look at patterns and as follow-ups, the problems are extended to larger denominators like $^{13}\!/_{20} + {}^{4}\!/_{20}$, $^{2}\!/_{15} + {}^{4}\!/_{12}$, $1\frac{7}{8} - 1\frac{1}{3}$

Student Evaluation: Magic Square with fractions

Students were supposed to work in pairs and fill up the empty boxes with fractions that will make each row and column equal to 1. They are not supposed to use the same fraction more than once.

Lesson Plan of Second Lesson Observed

Name of Teacher: Hajah Nuzailah
Class: 1E (Year/Grade 7)
Topic: Subtraction of Fraction.

Learning Objective: So that children are able to think mathematically how to solve problems related to subtraction of fractions (like and unlike fractions).

Other Objectives: 1) For students to be interested in the lesson (have the right attitude to involve in mathematical thinking)
2) For students to think mathematically in integrating what they know and relate it with what they are currently learning and to make sense of the lesson using the manipulatives provided.

Introduction: Is there enough Pizza?

Jamil bought one large pizza and plan to share the pizza with his friend. He divided the pizza as follows:

Name	Amount of pizza
1. Jason	¼
2. Jill	1/8
3. Jamilah	3/8
5. Johari	2/5

Questions Posed	**Corresponding thinking**
8. If Jason is the first to eat the pizza, how much is left for the rest	Students are supposed to subtract a quarter from 1 to get three quarters.
9. If Jamil gave 1/8 of the remainder to Jill, what fraction is left?	Students are supposed to subtract 1/8 from three quarters
10. During break time, Jamilah ate 3/8 of the remaining pizza. How much pizza	Subtracting like fraction

is there left?	
11. Is there enough pizza for Johari? Is his share bigger or smaller than Jamilah?	To test students' understanding on order of fraction (which is bigger and smaller
12. What happen to Jamil's share? Is it getting more or less?	To make student think what happen during subtraction

Activity (Paper folding)

Students are given blank papers, and are encouraged to translate the questions using he rectangular paper to get answers (by folding and shading). Students were asked to look at patterns and as follow-ups, the problems are extended to larger denominators like $13/22 - 4/22$, $4/15 - 2/12$, $1\frac{7}{8} - 1\frac{1}{3}$. Although there was no question involving mixed number or improper fractions above, children were asked one question involving these fractions during the activity.

Student Evaluation: Worksheet on subtraction of fraction

The questions on this worksheet were the normal exercise book problems that we often see.

Appendix B (Video Description)

Title: Addition and Subtraction of Fraction

Teacher: Hajah Nuzailah Haji Nali

Class: Form 1A and 1E (Grade 7)

School: Sekolah Menengah Masin

Date: 1st lesson – 14th April, 2007 for class 1A
2nd lesson – 24th April, 2007 for class 1E

Team member: Dr Madihah Khalid (UBD), Mohd. Khairul Amilin Haji Tengah (UBD), Dr Zaitun Hj Taha (UBD), Mr Masaaki Takahashi (MTSSR), Mr. Palanisamy Veloo (CDD)

Introduction

The first lesson was on the topic of addition and subtraction of fractions. The teacher planned the lesson very well, laboriously prepared the teaching aids, determined fully the procedures of her teaching and prepared interesting activities. Everything was written in her lesson plan as shown in Appendix A. Since the aim of the lesson was to incorporate mathematical thinking into each lesson, she has also prepared to ask appropriate questions to students directly to elicit students reasoning.

In the introduction of the lesson, the teacher tried to motivate the students by telling a story so that children would be interested to learn and be involved in class. This is one aspect of mathematical thinking, (the mathematical attitude) that the teacher is trying to address. The names of the characters in the story were adapted from the names of famous people. She probes and asks students questions but usually gets chorus answer. She could have improved her questioning techniques if she set some rules like whoever wants to speak in class should raise their hands-up and she would in turn call out the children. From our discussion, we think that she should have asked the students why "when it comes to addition and subtraction, the denominator need to be the same".

Activity (Paper folding)

Paper folding was used to help students translate numbers and symbols to the concrete form for better understanding. Transferring round pizzas into rectangular papers also needs imagination and is also another aspect of mathematical thinking. It is also here that the teacher should stress the importance of equal parts, to allow these parts to be added or subtracted together. Maybe, ask one of the students to explain this.

Some students are not really interested in folding papers since they were already taught the addition/subtraction algorithm. When faced with addition or subtraction of unlike fractions, they would always suggest finding the "LCM" or "least common multiple". I found that children don't really know what LCM really is. They just know how to find them. Even when asked "What is the LCM of 2 and 4, they insist on performing the division" (algorithm).

Children are still not communicative enough and this is where a teacher's skill in probing would be handy. In my opinion, students in Brunei are still not participative enough in the class and this lack of communication and the inability for them to explain their thoughts make it a challenge for teachers in Brunei in reading their thoughts (mathematical thinking).

Assessing students' understanding
Teachers are supposed to encourage students' self evaluation by asking right questions. However, only a few pupils could explain well. Therefore, to further assess their understanding, students were asked to work in pair to work out the answers to the magic square with fractions. Different pairs of students were given different number combination. An example of one of the combination given is as follows:

	$^{10}/_{20}$	$^{3}/_{9}$
$^{4}/_{12}$		
		$^{6}/_{16}$

Children were found to still struggle to complete this simple exercise and they could not finish it in class because time was up. They were however encouraged to complete it at home and bring to the next lesson. During discussion with the teacher, she was advised to just concentrate on addition in one lesson and subtraction in another lesson, because one hour is too short for both.

The Second Lesson

The second lesson that the team observed was a lesson subtraction of fraction with class 1E. These pupils had a lesson on addition of fraction five days before and the team could not observe the teacher.

Evaluation of the lesson
Again, to make the lesson interesting, names of students in the class were used. It aroused students' interest and stimulated their mathematical thinking. This lesson was better executed than the previous one and children enjoy it because they have enough time to think and do the activity with their teacher. However, there is still the tendency for pupils to solve problems using LCM. Although the teacher tried to make them think in terms of equivalent fractions, they still insist on solving it using LCM. I guess old habit dies hard.

LESSON STUDY AS A STRATEGY FOR CULTIVATING MATHEMATICAL TEACHING SKILLS
A CHILEAN EXPERIENCE FOCUSED ON MATHEMATICAL THINKING

Francisco Cerda B.
Ministry of Education, Chile

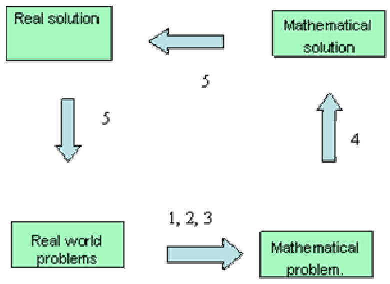
Introduction

In Chile, approximately 100% of school age children are registered in primary school. Nevertheless, the school attendance does not ensure that mathematics learning is of the expected quality. The testing results indicate this lack of quality in math learning. There is great interest in our society as a whole to focus on a qualitatively change in the practice of math teachers. Chile has been carrying out multiple initiatives in this sense for several years and lately it is participating in a project to improve the education of mathematics with the technical assistance of Japan. At the same time, within the framework of the collaboration between the economies of APEC, Chile participates in a collaborative project in mathematical education that contemplates a progressive study of different subjects related to mathematical education: good practices, mathematical thinking, communication, evaluation, and generalization towards other subjects. This report gives an account of an experience that, following the methodology of Lesson Study, was developed according to the didactic principles of Consultant's School Strategy for the Curricular Implementation in Mathematics, strategy developed in Chile. The comparative analysis of the two previous strategies, made by Gálvez (2006), *allows us to conclude that both constitute powerful strategies to improve the teacher practice and, at the same time, to generate processes of professional learning of the teachers, which guarantees a greater stability in the changes obtained by its performance*.

Lesson Study focused in the mathematical thinking.

To focus the processes of study in the development of the mathematical thinking implies a double challenge; it involves the students as much as math teachers. Professor Katagiri (2004) aiming at the autonomy of the students indicates: "*Cultivating the power to think independently will be the most important goal in education from now on, and in the case of arithmetic and mathematical courses, mathematical thinking will be the most central ability required for independent thinking*".
On the other hand Stacey, K. (2006) emphasizes that "*providing opportunities for students to learn about mathematical thinking requires considerable mathematical thinking on the part of teachers*"

A key window considered in Chile to develop the mathematical thinking mentions the mathematization of real world phenomena. The cycle of mathematization has been described by means of the following scheme: (Jan de Lange, 2006):



1. Problem situated in reality
2. Identification of the relevant mathematics
3. Gradually trimming away the reality
4. Solving the mathematical problem
5. Interpret the meaning of the mathematical solution in terms of real world.

To connect a problem of the real world with mathematics is not trivial and requires a fundamental mathematical competence. According to (OECD-PISA), today at least eight key mathematical competences are distinguished, that allow this connection; they are: to think and to reason, to argue, to communicate, to model, to create and to solve problems, to represent, to use language and symbolic operations, formal and technical and, to use aids and tools.

The Consultant's School Strategy for the Curricular Implementation in Mathematics (LEM), sustained by the Ministry of Education in alliance with Universities of the country, is based on the following didactic principles for the learning of mathematics (Espinoza, 2004)

1. In order to learn, the student must take part significantly in the mathematical activity, and not only limit him/herself into accepting and applying the strategies taught or “shown” by the teacher.
2. Learning consists of a change of stable strategy, the replacement of knowledge by another one, caused by an adaptation to a situation.
3. The mathematical knowledge arises from the work of the children as the optimal answer to specific problematic situations that require it.
4. The learning activities must constitute true challenges for the children, when putting in crisis/conflicts their previous knowledge. These activities must be accessible to the children and have their frame of reference in familiar and significant contexts.
5. When the teacher (or the text) gives the necessary instructions to do the task correctly, it is the teacher who is using the required mathematical knowledge and not students.
6. The mathematical knowledge in a learning process must appear as the necessary knowledge to pass from the initial strategies, low efficient and inadequate, to the optimal one.
7. The students choose and share different resolution techniques. The “error” is a substantial part of the learning process.
8. The knowledge and mathematical procedures used must be valued by all the children. It is recommendable not to spend a long time between the moment at which the mathematical knowledge has emerged for the children, and the moment

at which the teacher emphasizes and systematizes it. But, it does not have to be formalized prematurely.

9. The students must have the opportunity to work and to deepen the knowledge until obtaining a significant dominion of it.
10. The argumentation and mathematical explanation are seen as laying the basis for the adjustment of the algorithms and the modification of the mistakes.

Description of a Lesson Study experience in Chile focused on Mathematical Thinking

This study took place in the Municipal School *Dr. Luis Calvo M.* (Santiago), but it had the participation of other teachers as observers, they belonged to two other schools (*R.I. School* and M, A.C. School), and they wished to become future developers of the designed lesson. The criterions to choose the school were, they have at least, one course per level, to have the conditions to develop an authentic experience, not to simulation, and the schools belong to Whole School Pilot Project (supported by the Ministry of Education and constituting part of Consultant's School Strategy for the Curricular Implementation in Mathematics). The process that is described in this work contemplated six phases that are described briefly next:

Phase 1 Presentation of Lesson Study Project to the group of invited teachers.

The first conversation was carried out in the school Luis Calvo M. and was attended by the mathematical teachers of that school plus the invited ones. They observed a PPT with the foundations of the Collaborative APEC Project in Chile, whose long term goal is to promote the development of mathematical thinking from the perspective of the mathematization processes. The author of this report assumed the role of external adviser, summoning the group and carrying out the presentation of the project. They discussed the following text (Takahashi, A., 2006): *the idea of lesson study is simple: collaboration with fellow teachers to plan, observe and reflect on lessons developing a lesson study, however is a more complex process Because lesson study is a cultural activity, an ideal way to learn about lesson study is to experience it as a research lesson participant. In so doing, you will learn such things as how a lesson plan for lesson study is different from a lesson plan that you are familiar with, why such detailed lesson plan is needed, what type of data experienced lesson study participants collect, and what issues are discussed during a post lesson discussion*.

One of the teachers stated that this group had elaborated, last year, a Didactic Unit (from now on DU) of Geometry in a course of teacher training and that had been only applied by one teacher, in October, 2006, but the other designers could not observe it. This DU was directed to 8^{th} grade; its subject was the area and perimeter of circumferences and consisted of at least eight classes. Since the subject of geometry is of particular interest, they agreed to start off with a critical revision of this DU before carrying out a new design. They agreed, in addition, to invite the teacher who had applied the Didactic Unit with the purpose of knowing first hand, how was this process. He was given a task of reading and of studying that Didactic Unit. Eight sessions were set up as a minimum to develop this project. All the assistants valued the opportunity to meet and reflect on the education of mathematics. They said that they had the idea to form a reflection group on mathematical education, that they would call *Pythagoras Group.*

Phase 2 Critical revisions of the subject area and perimeter of a circle
Teacher C.M, who worked on this subject, told them how he lived the experience when he applied the didactic unit designed by them. In particular, the discussion focused towards the second class that corresponds to the characterization of the number π. The class contemplated three moments:

- Measurement of the contour of circular base objects, using a ruler or measuring tape.
- Searching for a broad approach of how many times the diameter fits in the circumference (arriving at that it is 3 times and "something more". The lacking difference is not quantified.
- from measurements of the contour **P** of a circle and its diameter **d**, calculates the reason: **P/d**

The following photos illustrate the moment at which the students find that the diameter fits at least three times in the contour of the circumference. In doing this they bordered with plastilina the contour of the circumference.

There was an intense and interesting discussion that was centered in the following key points:

- Why the students did not quantify how 3 diameters need to cover the perimeter of the circumference? This course already has the tools to have done it (but, they only obtained value three).
- The plastilina had a problematic performance because it is tensile. Some children stretched it to fit it into the circle contour three times.
- When they did the quotient between the measures of the perimeter and diameter of the circumference (cm.), obtained numbers with three or more decimal, but those decimals cannot be considered as valid values; there exist a physical limitation in the measurement that prevents obtaining more precision than mm

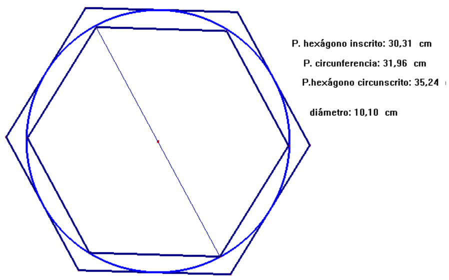
gives. In spite of this, the students obtained decimal values for the quotient, like the following ones.

	Perimeter cm	Diameter cm	Perimeter/diameter
Circle 1	25,1	8	3,1375
Circle 2	19	6	3,167
Circle 3	12,7	4	3,175

- The Teachers' team decided to focus their lesson design in the measuring of the contour of the circumference using as a unit of measurement its diameter and **quantifying** the magnitude of the remaining segment (smaller than a diameter).
- Another question that arose: which is the better moment for presenting the number π? Since it is very common in present education that π is defined early as the quotient **P/d.** They thought it was not necessary a premature definition of this number, it is better to obtain first an approximate experimental value and then to characterize it in a stricter form.
- One of the teachers was requested that he simulate a class about this same subject matter, but under the traditional way. He did it, and then an interesting discussion took place around the differences in the mathematical thinking that is put into play in each modality. The first proposal of a class was sketched and a task was given: advancing in the design of the class in individual form.

Phase 3. Design the research-class.

The design of the research-class begins. It will center in obtaining a quantitative relation between the perimeter of a circle and its diameter. For this purpose they will set out a sequence of measurements of the contour of a disc, varying in each case the measurement unit that is used: initially without measurement instruments, then using a graduated ruler, continuing with a metric measuring tape, to culminate with the use of the diameter as measurement unit. Product of the previous process they will obtain a good approach for the number π.



Teacher A.F. raises the idea to complement later the work later of this class by means of an activity with the geometric software Cabri Géométre II that will consist in drawing a circumference with a inscribed and circumscribed hexagon in it, getting the respective measurements of the perimeters with the Cabri tool for measure, and comparing those values with the experimental values obtained in the designed class. Also he raises the idea that the students analyze a documental about the interesting history of the number π. There remains a task: to write up a draft of the class plan.

Phase 4. To complete the lesson, establishing the necessary materials.

The intentions of this session are: to raise the hypothesis about what will happen in the planned class, to prepare good questions for the students, to anticipate possible ways of solving the problems, to raise and to assign different tasks about the observation of the participants. They agree that the students will work in small groups and will have discs on which they will carry out the measurements. Each professor brought different circular objects. They decided on covers of round plastic containers. The materials and the activities are proven. When carrying out the quantification of how many times the diameter d fits in the perimeter P of the circumference, they obtained an approximated value of $3\frac{1}{9}$, of the reason P/d. The used technique consisted of the segment that exceeds 3d was marked on the paper tape (of length d) and then this one was bent successively. They checked that the segment fitted 9 times. They defined a sequence of two classes: In the first class the contour of a disc will be quantified, but in the course of the lesson the conditions of that quantification will change in a progressive form: they measure first with any no conventional unit, then with a ruler, later with a metric tape and it will be culminated with a **measurement that uses the diameter** as the unit of measurement. One hopes that the students quantify the leftover segment after applying 3 times the diameter in the contour of the circular object. That quantification, according to the previous knowledge would have to be expressed through a fraction of the type P = $3\frac{a}{b}$ times the diameter.

Phase 5 the lesson was taught

The two designed classes were applied in a consecutive form and both were observed by teacher R.L of the same school. In addition the lesson was recorded by the knowledgeable other. This report will be centered in the analysis of class 1 only.

Phase 6 Discussion after the lesson implementation.

The objective of this phase is to put in common the observed things, as well as the performance of the teacher and the answers of the students, to propose ideas to improve the class plan and to make decisions with respect to giving continuity to the joint work. The group of teachers evaluated class 1, through the observation of a video clip of 30 minutes. From that reflection there were the following conclusions:

- In general the class was developed with fluidity and they noticed that the students were involved in it.
- The closing of the class was missing with questions directed to the students, nevertheless the professor made many questions to the groups but not to the entire course.
- With respect to the lived experience, the teacher who led the class indicated: "*I could not always teach classes with this same modality. As an activity for introducing a subject, I agree, I believe that the richness of the learning comes from this, but gradually or at some time it is necessary to return to the role of the "demanding mathematical teacher", because they are going to face new texts, new institutions, other teachers and they have to be prepared*."

Another teacher responded to him: *"We are looking for the participation of the students in the construction of new learning. What happens in the traditional classes is to start off giving the value of π and the formula of the perimeter of the circumference. Here everything is oriented to the origin of π and to find out that the circumference perimeter is independent of the circumference size. If we understand that, we have made a significant advance in the understanding of everything that comes ahead."*

- About the role of the observer: he must have a more active role in the gathering of information. The observer teacher indicated that he felt a desire to interact in the class with the students by asking them some questions. In fact he did it more than once. It was recognized by the group that for a research-lesson like this one, it is very difficult for the teacher of the course to pay attention to everything that happens in the classroom.

In relation to mathematical aspects, they remarked:

- It is necessary to focus on having the students understand that there is a (liner) dependency between the perimeter of the circumference and its diameter. It was stated that there is not a good estimation of which is the measure of the contour of a circumference; in general, people think that it is less than the real measure. A teacher remembered that in certain professional activities they used a metric wheel that in each turn measured 1 meter, and asked the other teachers. Which is the radius of that wheel? And the answer took some time.
- In a another group, when measuring the contour with the diameter, one student said, "it fits 3 times and exceeds a small piece, but we cannot say that it fits 4 times". Another one said: it fits three times and 2 cm. (they mix the units: the diameter and the centimeter).
- With this work they arrived directly to an algebraic expression to find the perimeter of a circumference, it was the consequence of the previous process and it was necessary to formalize it. So far they had obtained a very good approach for π of $3\frac{1}{7}$
- The observer teacher indicated that the teacher missed insisting that in all cases it gives a constant value for the quotient P/d, independent of the size of the disc. *"There is not a little π for small circumferences, and a great π for large circumferences"*
- The teacher responsible for the class said: "I need the students to work with Cabri so that they see that π is a quotient comparison and it has the same value in any circumference", that is to say, π is a ratio. In relation to the question about whether these students have learned the concept of ratio, the answer is no, there is a need to discuss it further in order to see that this ratio is constant. The teacher concludes: he could review the subject about the meaning of π when the subject of proportions will be studied later in the class. It is necessary to arrive to other a problem in which π is used. Another teacher asked: In what other subject π is used? , and the external adviser indicated that in probabilities. There is an experiment of Buffon (century XVIII) that consists of throwing a needle of length

b, on a board divided by lines separated from each other by a certain distance ***a*** (smaller than ***b*** or just as ***a***). Knowing that the probability that the needle falls on one of the lines is 2b/ (a• π), so we can consider π like in the previous case. We can simplify the experiment taking ***a*** = ***b***, with which the probability of the event is 2 /π, and then dividing 2 by the frequency whereupon the event happens we will have our approach of π be the same result as when Buffon made the experiment throwing the needle and obtaining π until with 3 decimal numbers.

- When seeing the techniques that the students used to respond to the challenges raised in the class, we verified that there are some techniques that the teachers could not anticipate: "*When we thought about the class we did not imagine this.*" For example with the use of the ruler: that they were going to draw the circle on the paper and soon they would mark cords of 1 cm, one after the other. In the case of an 18 cm diameter circle, they managed to mark 56 cords of 1 cm, that is to say, they registered a regular polygon of 56 sides. The result has an error of less than 1%; we remembered that with a regular hexagon the error registered is of ≈ 4.5%.

Relative to the performance of the students

- In a group, when a student was asked to explain what he obtained, he refused by saying: "*I cannot express concepts with words*". Teachers reflect on this point asking themselves if it is a language problem. One of them indicated that the student did not know, therefore he cannot explain. Another teacher answer him that he does not think the same, since he observed that the student participated actively in the class, thus he thinks that he lacks mathematical language to express what was experimented. Another teacher commented that one of the key competences in mathematics is communication and that it is neglected in the mathematics class.
- The teacher who led the class indicated that his evaluation of the lesson was positive, he was impressed by the children movement in the room, the spontaneous discussion and the speech of each one of them: "I felt them to be true protagonist of their own learning."

ANALYSIS OF THE LESSON

The following analysis is taken from the Anthropological Theory of the Didactic. (Chevallard, 1997). The subject treated in this class can be considered like an isolated mathematical organization conformed by a mathematical task, the techniques that allow making this task and by a theoretical speech that allows us to explain the techniques and to give a theoretical sustenance to them.

Mathematical task of the class:

- To quantify the perimeter of a circle.

Didactic variables:

- Availability of a circular object (disc) in class 1 or the drawing of a circumference on the paper. (in class 2)
- the type of measuring instrument available to carry out the measurement, or the unit of measurement to be used in the quantification (example: the length of the diameter)

Conditions: these vary progressively throughout the class:

- does not have measurement instrument, only have the disc
- use of one ruler,
- use of a metric tape
- has only a measuring tape of equal length to the disc diameter.

Techniques:

- Uses pieces of paper like a unit of measurement
- Uses its fingers like unit of measurement
- Marks a point of the circular object and makes it roll on to the ruler until the marked point returns to the initial position.
- They copy the contour of the circumference in a sheet of paper and with a ruler they measure with cords of 1cm drawn up one after the other.
- Turn the ruler around the disc. They border the circular object with the paper metric tape measurer

- Put the tape on top of the diameter in the contour concluding that it fits 3 times, but it exceeds a segment smaller than the diameter.
- to quantify the segment that is left successively doubling by half the unit of measurement and obtain the fraction $\frac{1}{8}$ of the diameter as an approximate value (division of the unit)
- Cut a tape to the length of the leftover piece and successively put it on top of the diameter to find out how many times it is possible to fit it (repeat the measure of the piece as many times as it is necessary to cover the diameter completely).

In the development of the class three essential moments can be distinguished:

Beginning Moment: When the mathematical task of this class is presented and it consists of measuring the contour of a circular object. The students react at this moment, according to the conditions put forward by the teacher, without any conventional measures (they measure with the fingers or a piece of paper).

Development Moment: Starting from a progressive change of conditions (availability of ruler, tape, the diameter as unit of measurement) the students elaborate other techniques of resolution. Here the students are faced with diverse obstacles according to the instruments they use, if it is a ruler they have the problem of measuring a curved line with a straight instrument, a thing that is avoided when they use a metric tape, because this one can take the form of the circular object that is being measured. The culminating moment arrives when the students are asked to quantify the contour of the disc only using a piece of paper of equal length to the diameter of the disc. In the first approach, all obtain the result that the diameter fits three times in the contour, but exceeds a small arch that must be quantified when they only have a unit of measurement greater than the length of that arch. Thus, it is a problem, because normally one measures with units smaller than the object to be measured. This implies, necessarily, that the unit of measurement will have to be divided. It is here where diverse techniques arise to obtain the fraction of the diameter. Some students (successively) double the unit of measurement by half until they obtain the eighth, verifying that this fraction of the diameter is very close to the length of the leftover arch. Others mark, on the unit of measurement, the length of that arch and soon they double (successively) according to that measurement, to obtain, either a $\frac{1}{6}$, a $\frac{1}{7}$ or a $\frac{1}{9}$.

Closing Moment: the proposed tasks are reviewed and the techniques used are compared. The new knowledge is identified and institutionalized: the diameter of the circle (measured in cm.) multiplied by some of the following fractional numbers can obtained good approaches to the perimeter of a circumference by multiplying: $3\frac{1}{6}$, $3\ \frac{1}{7}$, $3\frac{1}{8}$, $3\frac{1}{9}$, $\frac{1}{7}$ being the best approach, since it gives the first two decimals in an exact same form: 3,14. The class culminates with a verification activity that consists of: each group must designate a member to go to the blackboard (adherent paper tape exactly to the length of the perimeter of their disc. Then he returns to his group and verifies that it is the right measurement to cover exactly the contour of the disc, without lacking or going over adherent tape. The group that does it well, wins. The class was called: "**How much does a wheel move in one turn?** The students learned to quantify the length of a complete turn of a wheel. From the point of view of modeling (one of the key mathematical competences), we can say that in this real world problem a mathematical structure was imposed: $P = 3\frac{1}{7} \bullet d = \pi \bullet d$, formula that relates the perimeter of the circumference with the length of the diameter.

Final Reflections, Conclusions and Projections

Having finalized this first part of the project we can indicate that it had many benefits for all the participants. The students were committed and enthusiastic with the work proposals; they enjoyed the activities, discussed among themselves and with the teacher the mathematical topic of the class. For the teachers it meant having a longed for space for dialogue and reflection about math education, not only with teachers of the host school, but that with teachers of two other schools. In relation to the institutional aspects, it is necessary to emphasize the strong support of the Director and the Technical Pedagogical Sub-Director of the School Dr Luis Calvo Mackenna, since they gave all the facilities to find the space and the time for meetings of the team that designed and tested this lesson study.. This reminds us of what a director of a North American school stated:

"If we are serious in the fight to constantly elevate the quality of the teachers we must provide to the teachers the time and the resources to them that they need to form a

School	Mathematical teachers who will implement the lesson	8th level courses	Students
Luis Calvo Mackenna	2	2	70
República de Israel	2	2	70
Miguel Ángel Cruchaga	2	3	120
Total	**6**	**7**	**260**

community that invests as much in the learning of the children as of the same teachers. If we are serious in not leaving no child behind, then we must provide a process so that the teachers assume the responsibility of a growth and continuous improvement that does not leave no teacher behind either”. (Liptak, L. 2005)

Limitations

- Little time available for the teachers to meet, in order to plan and reflect. They have too many hours of class teaching in their contracts.
- Difficulties to carry out the observer role, for the same previous reasons. To observe the research-lesson means to be absent from their own classes.
- Necessity of a “knowledgeable other”. At least during a period of time the intensive support of an external adviser person to the school is necessary. This support should probably go away smoothly until the participants gain experience and autonomy.

Projections

The host school is thinking to apply this research-lesson in another parallel class (8° B). The teachers of the schools that also participated as observers will apply this class in their respective classes; for this purpose a meeting will be held also with schools that did not participate in the project, but which are interested in applying it. The sustenance of this methodology has a good perspective in this school, since there is a supporting principal, depending on the disposition of the teachers to continue practicing it. The idea exists to invite teachers of the neighboring schools to participate in a session in which the class with real students is demonstrated and all the previous ones within the framework of a plan that can soon be discussed with them (LS Open House) to improve the learning of geometry. The following picture shows the quantitative projections of the application of this class:

Finally we wish to indicate that research demonstrates, (Nordenflycht, 2000), that the effectiveness of the changes in the educational practices is related to the level of participation of the involved ones in the process, of the degree of depth of the reflection and the analysis that they carry out in their own practice. On the other hand, the development of a common project requires the rupture of the isolation in which the teacher develops its professional work. This isolation in the professional exercise constitutes, without doubt, a brake for the generation of a collaborative work that is one of the foundations of professional development. Consequently, a proposal to improve has to make it possible for teachers to reflect on their own practice, a collaborative work in which the investigation and the innovation are closely bound in their role as guides and promoters of learning. It implies, in addition, to develop competences and strategies, to analyze and to interpret situations, and to promote viable and effective solutions and alternatives of qualitative improvement. An independent teacher is a subject able to carry out a design on his own, able to interpret his reality and its context, to take initiatives, in synthesis, a constructor of innovations.

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APPENDIX

General Information

Title: How much does a wheel move in one turn?

Topic: Perimeter of a circumference obtained from measuring the contour of a disc with a tape whose length is equal to the length of his diameter

Producer: Ministry of Education, CHILE.

Video recorder and video editor: Francisco Cerda Bonomo

Teacher: Alejandro Flores

Research Team: Rafael León, Alejandro Flores, Soledad Cortes, Christian Méndez

Knowledgeable Other: Francisco Cerda B

Collaborator: Ms. Grecia Gálvez

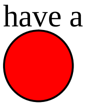
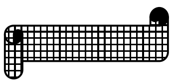
Host School:: Escuela Municipal Dr. Luis Calvo Mackenna. Santiago
Principal: Sr. Patricio Morales Borbal
Pedagogical Coordinator: Ms. Carmen Corvalán Fernández

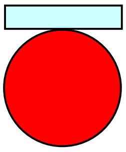
Invited Schools: Escuela Miguel Ángel Cruchaga, Puente Alto. Santiago
Escuela Municipal República de Israel, Santiago

Grade: 8th year of Primary School

Date: June, 2007

1. LESSON PLAN

Lesson 1	**Conditions** (teacher vary progressively throughout the class)	**Techniques** used by the students who allow them to make the mathematical task under specific conditions	**Remark**
Mathematical task to quantify the perimeter of a circumference	▪ They do not have measurement instrument, they only have a plastic disc 	• Use its fingers like unit of measurement • Use a piece of paper like a unit of measurement	Ask to the students: ▪ what value you consider it will have the length of the contour? Write down it! ▪ What difficulties present the measurement with a ruler? ▪ What differences you consider exist between the measurement with a ruler and with a paper tape?
	▪ use a graduated ruler (in cm and mm) 	• Marks a point of the circular object and makes it roll on to the ruler until the marked point returns to the initial position. • They copy the contour of the circumference in a sheet of paper and with a ruler they measure with cords of 1cm drawn up one after the other. • Turn the ruler around the disc.	
	▪ use of a metric tape 	• They surround the disc with a metric tape	

	Using a piece of paper of equal length to the diameter of the disc. Students will be requested to quantify the magnitude of the segment that exceeds 3 times the diameter. (This remaining segment is smaller than a diameter). 	• Put on top of the diameter in the contour concluding that it fits 3 times, but exceeds a segment smaller than the diameter. • to quantify the lacking segment successively doubles by half the unit of measurement and obtains the fraction 1/8 of the diameter like approximate value (division of the unit) • Cut a tape of the length of the leftover piece and successively put on top it in the diameter to find out how many times it fits is possible (to repeat the measure so many times of the piece as many times it is necessary to cover the diameter completely).	What fraction of the diameter is the leftover segment? They will be asked for to cut a piece of tape equal to the contour length of the disc.

2. EXPLANATION OF VIDEO

- **Title of VTR**: *"HOW MUCH DOES A WHEEL MOVE IN ONE TURN"?*

- **Summary**

This video shows a lesson for the eighth course of primary school. The study subject talks about the quantification of the perimeter of a circumference from the measurement of the contour of a circular object. From the conditions that the teacher is putting for the measurement, a progressive elaboration of techniques on the part of the students takes place. The central activity consists of which they measure the perimeter of the circular object using a tape whose length is the measure of the diameter of that object. One hopes that the students quantify that measurement using the whole unit of measurement and a fraction of it

- **Components of the lesson and major events in the class**

In the development of the class three essential moments can be distinguished:

Beginning Moment: When the mathematical task of this class is presented and it consists of measuring the contour of a circular object. The students react at this moment, according to the conditions put forward by the teacher, without any conventional measures (they measure with the fingers or a piece of paper).

Development Moment: Starting from a progressive change of conditions (availability of ruler, tape, the diameter as unit of measurement) the students elaborate other techniques of resolution. Here the students are faced with diverse obstacles according to the instruments they use, if it is a ruler they have the problem of measuring a curved line with a straight instrument, a thing that is avoided when they use a metric tape, because this one can take the form of the circular object that is being measured. The culminating moment arrives when the students are asked to quantify the contour of the disc only using a piece of paper of equal length to the diameter of the disc. In the first approach, all obtain the result that the diameter fits three times in the contour, but exceeds a small arch that must be quantified when they only have a unit of measurement greater than the length of that arch. Thus, it is a problem, because normally one measures with units smaller than the object to be measured. This implies, necessarily, that the unit of measurement will have to be divided. It is here where diverse techniques arise to obtain the fraction of the diameter. Some students (successively) double the unit of measurement by half until they obtain the eighth, verifying that this fraction of the diameter is very close to the length of the leftover arch. Others mark, on the unit of measurement, the length of that arch and soon they double (successively) according to that measurement, to obtain, a $\frac{1}{6}$, a $\frac{1}{7}$ or a $\frac{1}{9}$.

Closing Moment: the proposed tasks are reviewed and the techniques used are compared. The new knowledge is identified and institutionalized: the diameter of the circle (measured in cm.) multiplied by some of the following fractional numbers can obtained good approaches to the perimeter of a circumference by multiplying: 31/6, 31/7, 31/8, 31/9, 31/7 being the best approach, since it gives the first two decimals in an exact same form: 3, 14. The class culminates with a verification activity that consists of: each group must designate a member to go to the blackboard carrying only a tape of equal length to the diameter of the disc of its group and he must cut an adherent paper tape exactly to the length of the perimeter of their disc. Then he returns to his group and verifies that it is the right measurement to cover exactly the contour of the disc, without lacking or going over adherent tape. The group that does it well, wins. The class was called: "How much does a wheel move in one turn? The students learned to quantify the length of a complete turn of a wheel. From the point of view of modeling (one of the key mathematical competences), we can say that in this real world problem, a mathematical structure was imposed on it: $P = 3\ 1/7 \cdot d = \pi \cdot d$, formula that relates the perimeter of the circumference with the length of the diameter.

- **Possible issues for discussion and reflections with teachers observing this lesson**

1. What may be the goals of this lesson?
2. How can we characterize the mathematics of this lesson?
3. How does the teacher view his students?
4. What are the characteristics of the classroom management of this teacher?
5. Is there more mathematics stakes in this problem of which the teacher should be aware?
6. What may be the learning outcomes and the follow-up for such lesson?

MATHEMATICAL THINKING IN MULTIPLICATION
IN HONG KONG SCHOOLS

CHENG Chun Chor Litwin
Hong Kong Institute of Education

Introduction

The development of multiplication is a progress that allows students to learn a lot of patterns and also mathematics structures. This paper is based on a research of classroom teaching in mathematics of primary 3 to primary 5 students in Hong Kong. The same set of questions used in a primary 3 to primary 5 classes (except the question on Pissa). With the design of different sets of worksheets on non-routine problems, the development of concept of counting, repeated counting, multiple with counting, using multiple and then the law of multiplication to solve the problems is discussed. The results inform us that children using multiplication as a tool to solve questions of combinatoric nature (such as number of different grid formed by using different number of colour etc), are more difficult to understand than we thought. And the jumping of the cognitive gap from repeated counting and addition to multiplication needs certain daily examples to act as correspondence in concepts formation.

The using of the designed worksheets helps students to develop their concept of multiplication and also mathematical thinking through the connection of concepts by overcoming cognitive gaps.

The set of question used

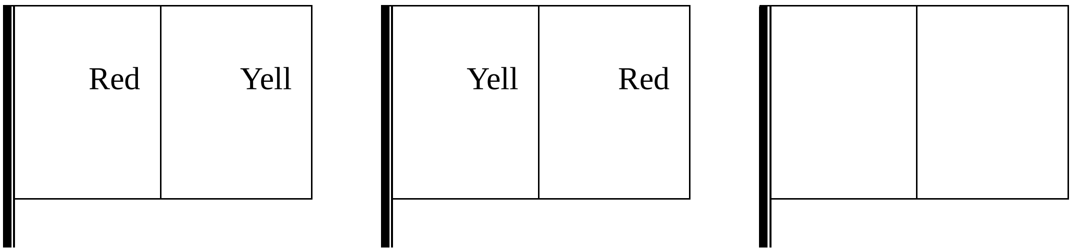
SET 1

Question A flag has 2-grid, if you have k colours to fill in the grids, and only each one of the colour is used once. How many different way are possible?

The worksheet has the following smaller questions. This is the format of all the sets of papers.

1	A flag has 2-grid, only each of the colours red and yellow can be used once. How many different way are possible?

2	A flag has 2-grid, only each of the colours red and yellow can be used once. How many different way are possible?
3	A flag has 2-grid, only each of the FOUR colours can be used once. How many different way are possible?



Students can find out that using FOUR colours can give them 12 different ways.

A	B	C	D	E
AB AC AD AE	BA BC BD BE	CA CB CD CE	DA DB DC DE	EA EB EC ED
5	5	5	5	5

By listing the table, some students are able to use the multiplication 5×4 = 20 to obtain the answer.
This is a systematic counting.
Can students jump from the results of 5 colours to 6 colours?
Many students still relay on the listing of the table for using SIX colours.

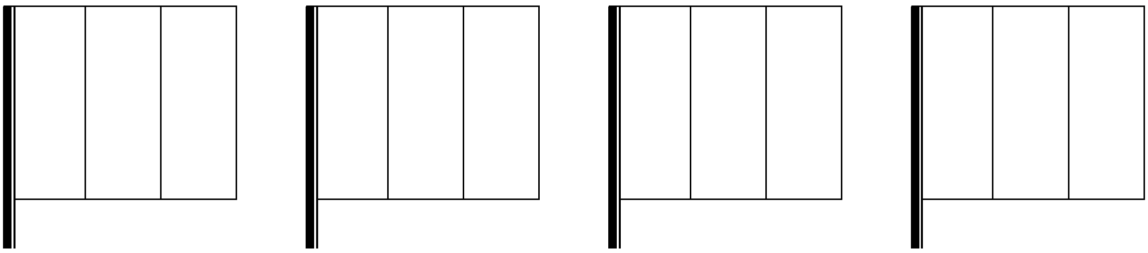
For primary 5 students, many can see that they have (k-1) choices for the first colour and k choices for the second colour. This is interesting that they think in the following process. If repeating the colours are allowed, for 7 colours, there will be 7×7 = 49 ways, but the answer is “(7-1)×7 = 42, for it is not allowed to repeat the colours”. They use (7-1)×7 = 42 rather than 7×(7-1) = 42. The same for the case of EIGHT colour, (8-1)×8 = 56 and not 8×(8-1) = 56.

Summary on 2-grids：	
Number of Colour	**Number of ways**
2	
3	
4	
5	
6	
7	
8	
k	

It is very difficult for students to generalize the case to “k” colours. Even for the primary 5 students, they are not able to see that the answer is (k-1)×k.
However, given a large value of k, say k = 100, some students can provide the answer of 100×99.

SET 2 using of 3 grids and k colours.

Question
A flag has 3-grid, if you have k colours to fill in the grids, and only each one of the colour is used once.
How many different way are possible?



Since there are three grids, the listing is more difficult than the worksheet in SET 1. Students start to use A, B, C to represent colours after some hints and they use listing to count the answer. This is difficult to use table and ABC to count 4 colours and the following is how some students tackle the problem.

For FOUR colours (A, B, C, D), the first colour A can give 6 options

A	B	C	D
ABC ACB ABD ADB ACD ADC			
6	6	6	6

Thy use counting for the first colour (A), and then they know that it will be the same for the other colour. Hence they got the answer 6 + 6 + 6 + 6 = 24.

And a while later, students formulate the results for 6×4 = 24.

Similarly, they do the same for FIVE colours in 3-grids. But still students could not obtain the relationship of 5×4×3 = 60.

However, many can fill in the following table up to case 8.

Summary on 3-grids：	
Number of Colour	Number of ways
3	
4	
5	
6	
7	
8	
k	

SET 3 Using number to fill in the grids

Question There are 3-grids. Insert the number 1, 2, 3, …, k into the grids once. How many different ways are possible?

The first question is using three numbers 1, 2, 3.

1	2	3

1	3	2

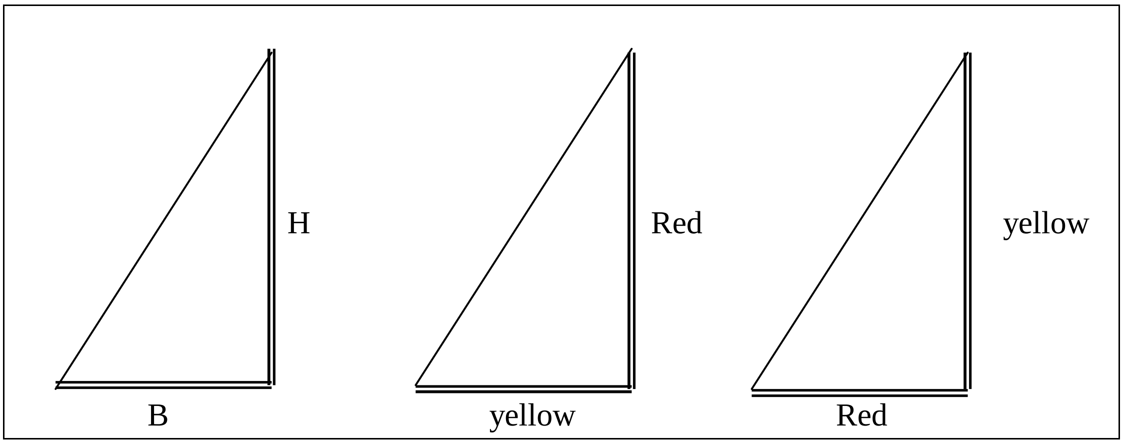
This set of papers is used in another class of primary 4. It is interesting that this problem, which does not have the context of a flag (grid), allow students to be more focus on the multiplication part.

The process of solution is similar. They list the cases and count the number. More students can obtain the results more quickly in compare to the answer in SET 2.

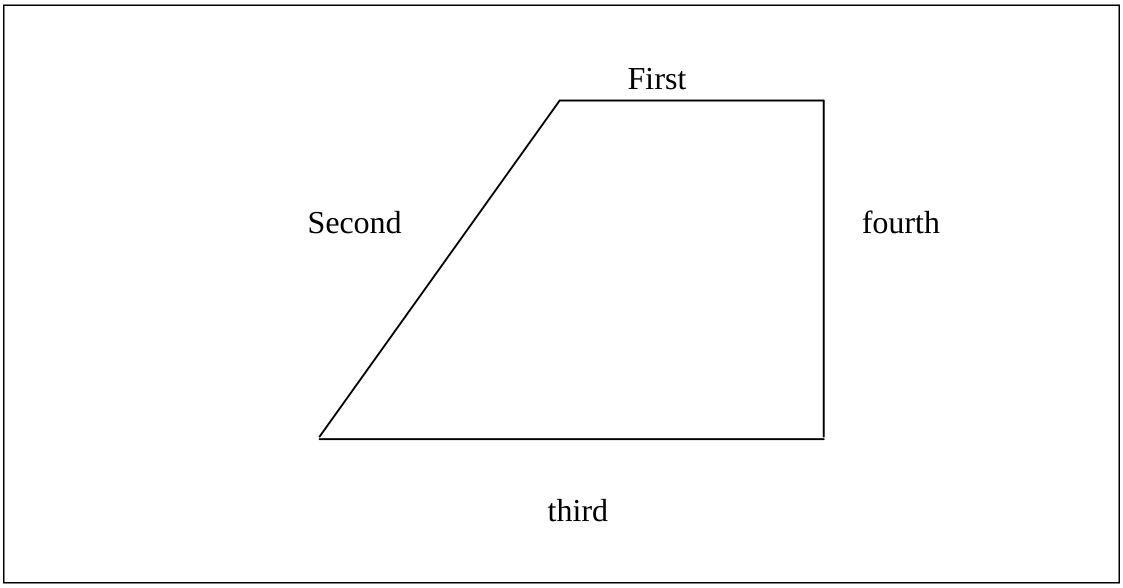
SET 4 Using a context of triangle

Question If you can only use the colours “red”, “yellow” and “blue” for the triangle, each side uses only one colour, how many different ways are possible?

Students use table to obtain the answer quickly for triangle and quadrilateral.



Ways	Base colour	Height colour
1		
2		
3		
4		
5		
6		
7		
8		



Quadrilateral	
Number of colour	Different number
4	24
5	120
6	360
7	840
8	1680
9	3024
k	(k-1)(k-2)(k-3)k

However, for pentagon, no primary 5 students can obtain the general for of the result (k-1)(k-2)(k-3)(k-4)k.

Pentagon	
Number of colour	Different number
5	120
6	
7	
k	(k-1)(k-2)(k-3)(k-4)k

SET 5 Pissa toppings

Question
You can choose a number of toppings for a pissa.
How many different kind of pissa are possible if there are k different choices toppings?

ways	Toppings		
	Cheese	Beacon	Sausage

For the case of two to three toppings, students can obtain the answer easily. They do that by counting. Students can use counting to handle the question up to 4 choices of serving. For 5 or more toppings, students' counting can be lost and correct answer can only be obtained through the using of multiplication.

For 4 toppings, many students can get 16 ways. Their strategy is by using ticks in the table.

Toppings	1	2	3	4
	✓	✓	✓	✓
	✓	✓	✓	✗
	✓	✓	✗	✗

Both primary three to primary five students use counting to solve the problem. However, after using three toppings, the counting process is tedious. For example, students make mistake in using four toppings and give out answer such as 15 or 12. They have missed some cases in their counting.

Those who counted correctly try to use the pattern of the first few answer to obtain the solutions. Students do think that answer must satisfy the pattern they discovered, that is, in the sequence of 2, 4, 8, 16, 32 etc.

The first thinking process is a recurrence relation. If there are 4 ways for two toppings, and when one more topping is added, then there are two possibilities for the case of two toppings. The first one is no third topping is needed in this 4 cases, and the second one is the adding of the third toppings for the previous 4 ways. So students based on 4 + 4 = 8 to get their answer and not based on $2^3 = 8$.

For example, the following is the answer of two toppings, 4 ways.

	A	B
1	✓	✓
2	✓	×
3	×	✓
4	×	×

Then they repeated the above pattern, adding the third toppings, by giving four "✓" and four "×"., getting the answer 4 + 4 = 8.

	A	B	C
1A	✓	✓	✓
2A	✓	×	✓
3A	×	✓	✓
4A	×	×	✓
1B	✓	✓	×
2B	✓	×	×
3B	×	✓	×
4B	×	×	×

They need the answer of two toppings to obtain the answer of 4 toppings, and the answer of 5 toppings to get their answer of 6 toppings etc. They know that the "second" answer is a double of the first answer. However, they could not directly get the answer of 6 toppings from multiplication.

However, some students found that they can formulate question into either taking each of the toppings or not taking the toppings. There are two choices for each topping and their results are summarized in the following table.

Toppings	1	2	3	4	5	6
Yes	✓	✓	✓	✓	✓	✓
No	✗	✗	✗	✗	✗	✗
Number	2	2	2	2	2	2

So he found that the answer is $2\times2\times2\times2\times2\times2 = 64$.

The totally abstract use of multiplication is based on the understanding of the problem, familiar with the multiplication table and also the relation of repeated addition.

Summary

The above teachings show that it is difficult for students to formulate the solutions through multiplication. The thinking process of addition, repeated addition and multiplication need certain space for students to overcome their cognitive gap. Through such discussion of the listing of table, students can think of different ways in solving the problem. Though some of their answer through counting is not correct, they can verify them after some classroom discussion.

LESSON STUDY ON MATHEMATICAL THINKING:
Developing Mathematical Methods in Learning the Total Area of a Right Circular Cylinder and Sphere as well as the Volume of a Right Circular Cone of the Indonesian 8th Grade Students

Marsigit, Mathilda Susanti, Elly Arliani

Yogyakarta State University, Indonesia

In this study, the researchers strived to uncover the aspects of students attempts in developing mathematical methods in learning the total area of a right circular cylinder and sphere as well as the volume of a right circular cone of the 8th grade students of Junior High School. The results of the research describe students' attempts in inductive thinking, analogical thinking, deductive thinking, abstract thinking, thinking that simplifies, thinking that generalizes, thinking that specializes, thinking that symbolize, thinking that express with numbers, quantifies, and figures. Students' mathematical methods can be traced through the schema of teaching learning activities.

INTRODUCTION

The Decree of Sisdiknas No. 20 year 2003 insists that Indonesian Educational System should develop intelligence and skills of individuals, promote good conduct, patriotism, and social responsibility, should foster positive attitudes of self reliance and development. Improving the quality of teaching is one of the most important tasks in raising the standard of education in Indonesia. It was started in June 2006, based on the Ministerial Decree No 22, 23, 24 year 2006, Indonesian Government has implemented the new curriculum for primary and secondary education, called KTSP "School-Based Curriculum". This School-based curriculum combines two paradigms in which, one side stress on students competencies while on the other side concerns students' learning processes. The School-Based Secondary Junior mathematics curriculum outlines that the aims of teaching learning of mathematics are as follows:

1. to understand the concepts of mathematics, to explain the relationships among them and to apply them in solving the problems accurately and efficiently.
2. to develop thinking skills in learning patterns and characteristics of mathematics, to manipulate them in order to generalize, to prove and to explain ideas and

mathematics propositions.

3. to develop problem solving skills which cover understanding the problems, outlining mathmatical models, solving them and estimating the outcomes.
4. to communicate mathematics ideas using symbols, tables, diagrams and other media.
5. to develop appreciations of the use of mathematics in daily lifes, curiosity, consideration, and to encourage willingness and self-confidence.in learning mathematics.

This is why the aim of mathematics education from now on is still urgently to promote mathematical method and to take it into actions. Above all, these lead to suggest that it needs to conduct classroom-based research to investigate the necessary driving factors towards students' ability to develop mathematical method.

THEORETICAL FRAMEWORK

Katagiri, S. (2004) insists that the most important ability that children need to gain at present and in future, as society, science, and technology advance dramatically, are not the abilities to correctly and quickly execute predetermined tasks and commands, but rather the abilities to determine themselves to what they should do or what they should charge themselves with doing. Of course, the ability to correctly and quickly execute necessary mathematical problems is also necessary, but from now on, rather than adeptly to imitate the skilled methods or knowledge of others, the ability to come up with student's own ideas, no matter how small, and to execute student's own independence, preferable actions will be most important. Mathematical activities cannot just be pulled out of a hat; they need to be carefully chosen so that children form concepts, develop skills, learn facts and acquire strategies for investigating and solving problems.

Mathematical method

Mathematical thinking has its diversity of simple knowledge or skills. It is evidence that mathematical thinking serves an important purpose in providing the ability to solve problems on one's own as described above, and this is not limited to this specific problem. Therefore, the cultivation of a number of these types of mathematical thinking should be the aim of mathematics teaching. Katagiri, S. (2004) lays out the followings as mathematical thinking related to mathematical method:

> inductive thinking, analogical thinking, deductive thinking, integrative thinking (including expansive thinking), developmental thinking, abstract thinking (thinking that abstracts, concretizes, idealizes, and thinking that clarifies conditions), thinking that simplifies, thinking that generalizes, thinking that specializes, thinking that symbolize, thinking that express with numbers, quantifies, and figures.

Questions for Eliciting Mathematical Method

Teaching should focus on mathematical thinking including mathematical method. Questions related to mathematical thinking and method must be posed based on a perspective of what kinds of questions to ask. Katagiri, S. (2004) indicates that quaestion must be created so that problem solving process elicits mathematical thinking and method. He lists question analysis designed to cultivate mathematical thinking as follows:

a. Problem Formation and Comprehension

1) What is the same? What is shared? (Abstraction)
2) Clarify the meaning of the words and use them by oneself. (Abstraction)
3) What (conditions) are important? (Abstraction)
4) What types of situations are being considered? What types of situations are being proposed? (Idealization)
5) Use figures (numbers) for expression. (Diagramming, quantification)
6) Replace numbers with simpler numbers. (Simplification)
7) Simplify the conditions. (Simplification)
8) Give an example. (Concretization)

b. Establishing a Perspective

1) Is it possible to do this in the same way as something already known? (Analogy)
2) Will this turn out the same thing as something already known? (Analogy)

3) Consider special cases. (Specialization)

c. Executing Solutions

1) What kinds of rules seem to be involved? Try collecting data. (Induction)
2) Think based on what is known (what will be known). (Deduction)
3) What must be known before this can be said? (Deduction)
4) Consider a simple situation (using simple numbers or figures). (Simplification)
5) Hold the conditions constant. Consider the case with special conditions. (Specialization)
6) Can this be expressed as a figure? (Diagramming)
7) Can this be expressed with numbers? (Quantification)

d. Logical Organization

1) Why is this (always) correct? (Logical)
2) Can this be said more accurately? (Accuracy)

RESEARCH METHOD

The study was aimed at promoting students to develop mathematical method in learning the total area of a right circular cylinder and sphere and also the volume of a right circular cone. The approach used in the study was descriptive-qualitative of Lesson Study in two classes: the 8th grade of Junior High School, class A and the 8th grade of Junior High School, class B. The design of the research included: preparation (PLAN), implementation (DO), and reflection (SEE). The instrument used for collecting data consists of questionnaire, interview, observation of the lesson, and VTR of the Lesson. The research was began with two series of discussions between teachers and lectures and followed by observing and reflecting two lesson activities in the class, as the following description:

LESSON PLAN I

Day	**:**	**Thursday, May, 24, 2007,**
Date	**:**	**07.00-09.00**
Junior High School	:	SMP NEGERI DEPOK II, Yogyakarta, Indonesia
Grade/Sem/year	:	8/Sem II/2007
Teacher	:	Siwi Pujiastuti SPd
Class	**:**	**A**
Number of Students	:	40
Standard Competency	:	To understand the characteristics of cylinder, cone,

		sphere and to determine their measures.
Basic Competencies	:	To identify the formula of the total area of right circular cylinder; to identify the formula of the area of sphere.
Teaching Scenario	**:**	1. Apperception 2. Developing concepts 3. Reflection and presentation 4. Conclusion and closing

VIDEOTAPED LESSON I Part A :

Aim : To identify the formula of the total area of right circular cylinder

Introduction:

Problem Formation and Comprehension

- Teacher let the students observe given model of right circular cylinder *(concretization and induction)*
- Teacher let the students identify the components of the right circular cylinder *(method of abstraction)*
- Teacher let the students define the concept of right circular cylinder *(method of abstraction)*
- During the whole class teaching process, teacher encouraged students' abstraction *(method of abstraction)*

Group Work and Discussion:

Establishing a Perspective

a. Students employed concrete model to search the total area of right circular cylinder *(employing concrete model to express the concept and induction)*
b. Students broke-down the model of right circular cylinder into its components: two congruent circles and one oblong. *(employing concrete model to express the concept and induction)*
c. Students learned that the height of right circular cylinder is equal to the width of its rectangle; and the circumference of the circle is equal to the length of rectangle. *(employing concrete model to express the concept and induction)*

Group Work and Discussion:

Executing Solutions

- Students learned that the lateral area of right circular cylinder is equal to the area of its rectangle. *(analogy of concept and induction)*
- Students learned that the total area of right circular cylinder is equal to the area of its rectangle plus the area of its two circles. *(analogy of concept and induction)*

Students' Reflection :

- Students presented that the lateral area of right circular cylinder is equal to the area of its rectangle i.e.
  LATERAL AREA = HEIGHT X CIRCUMFERENCE OF CIRCLE = $t \times 2\pi r$
- Some students needed to have clarification that the lateral area of right circular cylinder is equal to the area of its rectangle. *(logical organization, analogy of concept and induction)*
- Students presented that the total area of right circular cylinder is equal to the area of its rectangle plus the area of its two circles i.e.
  TOTAL AREA = LATERAL AREA + 2 (AREA OF CIRCLE) = $t \times 2\pi r + 2(\pi r^2) = 2\pi r(t + r)$

VIDEOTAPED LESSON I, Part B:

Aim : to identify the formula of the area of sphere.

Introduction:

Problem Formation and Comprehension

- Teacher let the students observe given model of Sphere (Concretization and Induction)
- Teacher let the students identify the components of Sphere (Abstraction)
- Teacher let the students define the concept of Sphere (method of abstraction)
- Teacher's explained the way to find the area of the surface of Sphere.

Group Work and Discussion:

Establishing a Perspective

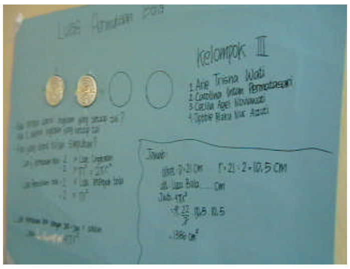
d. Students employed concrete model of a half Sphere to search the area of its surface *(employing concrete model to express the concept and induction)*

e. Students prepared the supporting facilities e.g. diagram of circles that has similar radius with the Sphere. (employing concrete model to express the concept and induction)

Group Work and Discussion:

Executing Solutions

- Students learned that the area of Sphere is equal to the area of its cover *(logical organization, analogy of concept and induction)*
- Students learned that the area of the surface of a half Sphere is equal to the double of the area of its circles. *(logical organization, analogy of concept and induction)*



Students' Reflection :

- Students presented that the area of the surface of a Sphere is equal to four times the area of its circles.
  AREA of SPHERE = 4 X AREA of CIRCLE
- Some students needed to have clarification whether their formula was correct? *(logical organization, analogy of concept and induction)*

LESSON PLAN II

Day	**:**	**Saturday, May, 26, 2007**
Date	**:**	**07.00-09.00**
Junior High School	:	SMP NEGERI DEPOK II, Yogyakarta, Indonesia
Grade/Sem/year	:	8/Sem II/2007
Teacher	:	Siwi Pujiastuti SPd
Class	**:**	**B**

Number of Students : 40

Standard Competency : To understand the characteristics of cylinder, cone and sphere and to determine their measures.

Basic Competency : To identify the formula of the volume of right circular cone.

Teaching Scenario :
1. Apperception
2. Developing concepts
3. Reflection and presentation
4. Conclusion and closing

VIDEOTAPED LESSON II

Aim : To identify the formula of the volume of right circular cone.

<table>
<tr>
<td></td>
<td>Introduction:
Problem Formation and Comprehension
- Teacher let the students observe given model of right circular cone. (concretization and induction)
- Teacher let the students identify the components of the right circular cone. (method abstraction)
- Teacher let the students define the concept of right circular cone. (method abstraction)
- During the whole class teaching process, teacher encouraged students’ abstraction (method of abstraction)</td>
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<table>
<tr>
<td></td>
<td>Group Work and Discussion:
Establishing a Perspective
f. Students learned the teacher’s guide to understand the procedures how to search the volume of right circular cone. (employing concrete model to express the concept and induction)
g. Students observed and manipulated the concrete model of right circular cone and right circular cylinder (employing concrete model to express the concept and induction)</td>
</tr>
</table>

<table>
<tr><td></td><td>Group Work and Discussion:
Executing Solutions
- Students collected data of the measurement of the volume of cone in comparison with the volume of cylinder. (logical organization, analogy of concept and induction)
- Students learned that the volume of right circular cylinder is equal to three times the volume of right circular cone. (logical organization, analogy of concept and induction)</td></tr>
<tr><td></td><td>Students' Reflection :
- Students presented that the volume of right circular cone is equal to one third of the volume of right circular cylinder. (logical organization, analogy of concept and induction)
- Some students needed to have clarification whether their formula was correct (logical organization, analogy of concept and induction)</td></tr>
</table>

ANALYSIS OF DATA

a. Problem Formation and Comprehension

The students manipulated *Concrete Model* of the *Right Circular Cylinder, Sphere and Right Circular Cone* in order to identify its components. They performed *mathematical abstractions* when the teacher gave them some questions or when the teacher let them work in group. Some students defined the *concept* of *Right Circular Cylinder* as its *functions* in daily life e.g. *"A Right Circular Cylinder is the storage to keep something like pen, pencil, etc."* The teacher encouraged the students to perform *mathematical abstractions* of the *Concrete Model* of the *Right Circular Cylinder* viz. to indicate its components such as circles, height, and the radius of its circle. There were students who defined *a Sphere* by giving the example of daily life e.g. ball, tennis-ball, etc. Students' *abstractions* of Sphere resulted the investigation of its components i.e. the *radius* and *diameter*.

At the first step, most of the students defined the right circular cone through *characterization of its shape,* e.g. *"A right circular cone is a thing like triangle", "A right circular cone is composed by three dimensional triangle",* and *"A right circular cone is composed by many circles – the higher the circle the smaller it does".* There were many ways in which the students *idealized* the geometrical concept. They mostly confirmed the concept to the teacher and asked their mates. Sometimes they performed their *idealization* by commenting other works. Some students asked the teacher why the *lateral area of cylinder is equal to the area of its rectangle* and why the *volume of cylinder is equal to three times the volume of its cone.*

b. Establishing a Perspective

Working in group triggered the students to develop analogical thinking of mathematical concepts. *Analogical thinking* happened when the students perceived that finding the lateral area of *Right Circular Cylinder* is similar to finding the area of its rectangle; and, finding the area of *Sphere* is similar to finding the area of its surface i.e. covering its surface by twisting around the rope. In sum, the concepts of geometrical shapes are mostly perceived to be analogical with examples in daily life e.g. the *right circular cone was perceived* as a *traditional hat.* In performing their *analogical thinking* the students frequently used strategic terminologies such as *"similar to", "comparison with", "the example of",* and *"the function of".*

Most of the students perceived that the given task by the teacher as the cases that need special consideration. Therefore, most of them considered more seriously on the ways to find the formulas of the total area of right circular cylinder and the formula of the area of sphere as well as the formula of the volume of right circular cone. Some students still paid attention on the concepts of cylinder, sphere, and cone. There was a student who wanted a clarification on the form of circle bases of cone whether it is convex, concave or plane. After getting input from teacher or their mates, the students usually *considered special cases* including corrected the formulas and made some notes at their works. Some students *considered* the use of $\pi = \frac{22}{7}$ *or* $\pi = 3.14$. One student expressed that for the bigger radius, we use $\pi = \frac{22}{7}$ and for the smaller radius we use $\pi = 3.14$.

c. Executing Solutions

Students' *inductive thinking* involved *Concretization and method of abstraction* in the area of *Problem Formation and Comprehension.* When the students who known the certain concept, were paced to perform *inductive thinking* they tend to reconfirm their concepts. *Inductive thinking* was spread from the beginning activities to the ultimate accomplishment when the students were paced to do so. The students observed the given model of right circular cylinder and strived to identify the components of the right circular cylinder in order to define the concept of right circular cylinder (method of abstraction). Students' *inductive thinking* was also related to *establishing perspective* in which the students employed concrete model to search the total area of right circular cylinder and broke-down the model of right circular cylinder into its components: two congruent circles and one oblong.

There were some steps in which the students perform *inductive thinking*, as following:

Inductive thinking* of finding the total area of *Right Circular Cylinder:

Step 1: Observing the *Concrete Mode*l of *Right Circular Cylinder*
Step 2: Manipulating the model and learning the teacher's guide
Step 3: Drawing the components of the *Right Circular Cylinder* i.e. the bottom circle, the top circle and its rectangle.
Step 4: Determining the area of its components
Step 5: Adding up the total area

Inductive thinking* of finding the area of *Sphere:

Step 1: Observing *Concrete Model of a half Sphere*
Step 2: Manipulating the model and learning the teacher's guide

Step 3: Twisting around of the model with the rope

Step 4: Thinking inductively that the length of rope needed to twist around a half model of sphere determines the area of circle.

Step 5: Finding out that the area of a half sphere is equal to two times the area of its circle.

Inductive thinking* of finding the volume of *cone:

Step 1: Observing *Concrete Model of Right Circular Cone and its Right Circular Cylinder*

Step 2: Manipulating the model and learning the teacher's guide

Step 3: Practicing to fill up the *Right Circular Cone* with *sand* which was acted of measuring out by the *Right Circular Cone*

Step 4: Thinking inductively to find out the volume of *Right Circular Cone* compare with the volume of its *Right Circular Cone*

Step 5: Finding out that the volume of the *Right Circular Cone* is equal to one third of the volume of its *Right Circular Cylinder*

d. Logical Organization

Logical organization of mathematical concept happened in all context of mathematical method: *idealization, abstraction, deduction, induction and simplification. Logical organizations* of mathematical concept can be indicated from the following example of students' questions:

1. Why is the lateral area of cylinder equal to the area of its rectangle?
2. Why is the volume of cylinder equal to three times the volume of its cone?
3. What happens if we do not carefully cover the surface of the sphere in which we use the rope for twisting around?
4. Is it true that the area of the surface of sphere is equal to 4 times the area of its circle?

DISCUSSION

Krygowska (1980) in Bonomo M.F.C indicates that mathematics would have to be applied to natural situations, any where real problems appear, and to solve them, it is necessary to use the mathematical method. The knowledge, skills, and mathematical methods are the foundation to achieve the knowledge on science, information, and other learning areas in which mathematical concepts are central; and to apply mathematics in the real-life situations. This study uncovered that teacher has important role to encourage their students to develop mathematical methods. The students performed *mathematical method* when they found difficulties or when they were asked by the teacher. Most of the students reflected that they paid attention on the perfect of the *Concrete Model* of geometrical shape. However, their consideration on the perfect form of the models did not indicate that they performed *mathematical idealization* as one of mathematical method.

Student's reflection:

Researcher	:	What is the effect on your calculation of the volume if a *right circular cone* is made up from *"a very thick metal plat"*
Student	:	It's okey. No, problem.
Researcher	:	Compare to a *right circular cone* if it made up from *"a very thin metal plat"* How can you determine the volume?
Student	:	In "*"a very thin metal plat cone"* I can fill up *more sand.* But if the radius and the height of the cone are similar, they should have similar volume.
Researcher	:	So, what do you think about the different between the cone that made up from *"a very thick metal plat"* and cone that made up from *"a very thin metal plat"* ?
Students	:	Yes, it will be very different. I am sorry for my initial statement.

Meanwhile, David Tall (2006) states that success in mathematical thinking depends on the effect of met-befores, the compression to rich thinkable concepts, and the building of successive levels of sophistication both powerful and simple. In this research, one aspect of mathematical method i.e. s*implifications* happened when the students perceived that the concept of *right circular cone* is similar to the concept of triangle or circle. In this case, they *simplified* the concepts through manipulation of *Concrete Models.* They also performed simplification when they broke down the formula to solve the problems. They mostly simplified the concepts when they had got some questions from the teacher; or, when they worked in group. Ultimately, when the teacher asked for the students to write the results, the students got that: 1) the total area of *Right Circular*

Cylinder is equal to the area of its rectangle plus two times the area of its circles, 2) the area of *Sphere* is equal to four times the area of its circle, and 3) the volume of *right circular cone* is equal to one third of the volume of its cylinder.

The students developed *inductive thinking* when they uncovered that the height of right circular cylinder is equal to the width of its rectangle; and the circumference of the circle is equal to the length of rectangle. They continued to perform *inductive thinking* until they found the formula of the lateral area of right circular cylinder; the formula of sphere, and the formula of the volume of *Right Circular Cone.* Students' *schema of inductive thinking* seemed in line with Katagiri's claim that *inductive thinking* covers:

> 1) Attempting to gather a certain amount of data, 2) Working to discover rules or properties in common between these data, 3) Inferring that the set that includes that data (the entire domain of variables) is comprised of the discovered rules and properties, and 4) Confirming the correctness of the inferred generality with new data

In a different context, Stacey, K (2006) suggests that a key component of mathematical thinking is having a disposition to looking at the world in a mathematical way, and an attitude of seeking a logical explanation. While Katagiri, S. (2004) claims that students' logical actions include: attempting to take actions that match with the objectives; attempting to establish a perspective; and attempting to think based on the data that can be used, previously learned items, and assumptions. In this research, the aspects of *logical organization* of mathematical concept emerged after the students put into practice mathematical procedures in their group. However, there were evidences that it was difficult for the students to practice mathematical procedures. Students' inappropriate organization of mathematical procedures appeared when the students had difficulties in performing mathematical procedure into practice with *concrete mode*l. In searching the formula of the volume of right circular cone, there were some students who hesitated what to fill-up with sand. Should it be a cone or cylinder? In searching the formula of the total area of *a right circular cylinder,* there was a question from the student, why the total area is the result of addition and not the result of multiplication.

CONCLUSION

In this Lesson Study, the researchers had sought to uncover the picture in which the teacher strived to promote mathematical methods in learning the total area of a right circular cylinder and sphere as well as the volume of a right circular cone. The striking results of the study can be stated that *students' mathematical methods* can be traced through the schema of teaching learning activities as follows:

1. Problem Formation and Comprehension were emerged when the students:
    a. observed given model of right circular cylinder, observed given model of Sphere, and observed given model of right circular cone
    b. identified the components of the right circular cylinder, sphere, and right circular cone
    c. defined the concept of right circular cylinder, sphere, and right circular cone
    d. got questions and notices from teacher to search the concepts
2. Establishing a Perspective were emerged when the students:
    a. employed concrete model to search the total area of right circular cylinder, the area of sphere and the volume of right circular cone
    b. learned that the height of right circular cylinder is equal to the width of its rectangle; and the circumference of the circle is equal to the length of rectangle
    c. learned the teacher's guide to understand the procedures how to search the volume of right circular cone
    d. broke-down the model of right circular cylinder into its components
3. Executing Solutions were emerged when the students:
    a. tried to find out the lateral area of right circular cylinder
    b. tried to find out the total area of right circular cylinder
    c. tried to find out the area of sphere
    d. collected the data of the measurement of the volume of cone in comparison with the volume of cylinder

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MATHEMATICAL THINKING AND THE ACQUISITION OF FUNDAMENTALS AND BASICS

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1. Fundamentals and Basics in Mathematics Education

The initial report of the Central Education Council issued in 1996 proposed that the formation of the "power for living," whereby one learns and thinks for oneself, in an educational environment free of pressure is the basis of education in the 21st century. Subsequently, based on a number of additional reports, the final report of the Central Education Council was issued, and the curriculum created accordingly along school curriculum guidelines went into effect in 2002. Under this curriculum, educators are required to help pupils form the capacity to identify issues on their own and proactively solve problems amid a society undergoing rapid change. According to school curriculum guidelines, fundamentals and basics are the basis that supports the various living and learning activities of children. For example, they are the basis of daily living activities, various school activities, the continuous learning of mathematics, and future social and lifelong activities. The guidelines emphasize the systematic acquisition of fundamentals and basics, through repeated study if needed, in order to enable the smooth pursuit of such activities.

The above approach to fundamentals and basics suggests a large number of issues related to current education and study guidance in the field of elementary school mathematics, such as the following:

A. An approach that combines guidance that emphasizes fundamentals and basics and nurtures individuality by enabling children to learn and think for themselves
B. Fundamentals and basics should be understood simply as formal guidance in terms of knowledge and skills, but they should include the aim of guidance in terms of the ability to think, judge, express oneself, and so on.
C. Activities to acquire fundamentals and basics should be understood in terms of children achieving goals through the autonomous study of problem solving, and fundamentals and basics should be considered to include study methods and the ability to solve problems.

The formation of the ability to learn and think independently should be understood in terms of the fundamentals and basics that make up the core that develops the drive to learn independently and the ability to proactively adapt to changes in society, in other words, the "power for living." Naturally, guidance for such a learning approach should be thought of in terms of not only textbooks but also the entire school spectrum and activities. Here, we examine the ability to learn independently, as fostered through the study of elementary school mathematics in particular.

The development of children who learn independently requires, first of all, teachers who are sufficiently aware of the importance of such endeavor and rigorous consideration of the type of guidance required to this end.

Furthermore, the ideal form of mathematics classes to teach fundamentals and basics to pupils has been considered based on the above. Fundamentals and basics are not just achieved through a one-hour-a-week class but need to be acquired over many hours (throughout the school year or throughout the course unit). The study of elementary school mathematics is characterized by a spiral-shaped progression of various component areas, with each area having a distinct study style, and it is necessary to promote practical research in the aspects of fundamentals and basics as well as the ideal guidance approach for each area, the problem-solving process per unit time, and the interrelation and realization of learning fundamentals and basics. In this paper, the author considers an approach to mathematics particularly from the aspect of fundamentals and basics.

2. Approach to Mathematics from the Aspect of Fundamentals and Basics

The aims of elementary school mathematics in Japan are the acquisition of fundamental and basic knowledge and skills regarding quantities and geometric figures and, based on this, the cultivation of the basics of creativity—such as the ability to think about things from a number of different aspects and the ability to think logically—as well as the development of an understanding of the merits of investigating and handling phenomena mathematically and the attitude of applying the knowledge thus obtained to subsequent objects of study or daily life. These aims include the definition of the type of mathematics instruction to be realized in the classroom. In elementary school mathematics, the cultivation of the ability to solve problems has been promoted for the formation of "how to learn." For the type of problem solving aimed for in elementary school mathematics, what is expected is the formation of the attitude of developing, through the solving of problems, new ways of looking and thinking about things that can be shared with others in the class and, in this manner, improve one another and achieve self-growth. The formation of how to learn demanded by the Ministry of Education, Culture, Sports, Science and Technology has been achieved based on the school research theme of developing the ability to solve problems. In this sense, in Japan's elementary school mathematics, developing the ability to learn and developing the ability to solve problems are considered to be almost synonymous.

Normally, when learning problem-solving skills, pupils progress through five stages. From the viewpoint of the formation of how to learn, the fundamentals and basics to be taught at each stage are as follows:

(1) Problem setting
Stage at which problems for study are created through the introduction of course units, class periods, etc.
- Ability to set learning targets and set learning sequences
- Ability to grasp the aims of the teacher's lesson progression
- Acquisition of the mathematical way of thinking

(2) Problem solving
Stage at which pupils grasp the situation and understand what the problem is
- Ability to form one's own question from the problem (visualization of situation and formulation of questions)
- Ability to share information about the problem with classmates
- Ability to engage in analogical reasoning from previous learning

(3) Solution planning
Stage at which pupils determine the direction required for solving a problem that has been understood
- Ability to recall previous learning contents and experience (thinking in terms of units, etc.)
- Ability to make predictions (prediction of meaning of calculation and estimation of solution)
- Ability to attempt problem solving on one's own (reexamination of calculation method)
- Intuition ability

(4) Solution execution
Stage at which a solution to a problem is attempted based on the solution plan and a tentative conclusion is drawn
- Ability to execute a solution based on a plan
- Ability to utilize past learning experience and contents (deduction from past learning)
- Ability to express one's thoughts in a manner understandable to classmates (actions

of thought,
judgment, and expression)

- Ability to recognize differences between one's own and one's classmates' thinking and to clarify the essence forming the background of these differences
- Ability to look back on one's actions (functional solution)

(5) Solution study (including review)
Stage at which pupils adequately evaluate processes and achievements and clarify what they have understood and what should be further investigated

- Ability to sympathize with the views of others, going beyond differences in thinking
- Ability to study solutions and come up with better results (perception of integration and development, refining of solution, integration with one's past learning, and acquisition of knowledge and skills)
- Ability to make generalizations and developments (generalization of a line of thought, abstraction, and logical processing)
- Ability to compare what has been learned in class with one's own and one's classmates' thinking and to evaluate oneself (validity of estimation)

The fundamentals and basics at these stages involve many different aspects, such as interest-, motivation-, and attitude-related aspects; mathematical thinking-related aspects; thinking ability-; judgment ability-; and expression power-related aspects; and knowledge- and skill-related aspects.

This section describes the mathematical way of thinking. The mathematical way of thinking is considered to form the core of mathematical education and the base from which arithmetic and mathematics knowledge and skills are produced. Let us examine the approaches discussed below, focusing on school year development equivalents in relation to children's awareness of the benefits and their application thereof through the learning contents in each area.

(1) Mathematical approach related to content—concept formulation, principles and rules
(2) Mathematical approach related to method—thinking in a mathematical manner
(3) Mathematical approach as a logical way of thinking—thinking in a logical manner
(4) Mathematical approach as an integrated and developmental way of thinking—thinking in an
integrated and developmental manner
(5) Mathematical approach that promotes a mathematical sense and thinking ability—numerical sense, quantitative sense and graphic sense
(6) Mathematical approach that promotes a mathematical attitude—approach to problem solving

- Concept formation, principles and rules: Thinking in terms of units, thinking in terms of place value numeration, thinking in terms of correspondence, percentages, etc.
- Thinking mathematically: Idealization, encoding, simplification, formalization, compaction, etc.
- Thinking logically: Analogic thought, inductive thought, deductive thought, etc.
- Thinking in an integrated and developmental manner: Abstraction, generalization, expansion, etc.
- Sensory and thinking ability: Estimation, sense of volume, approximation, etc.
- Arithmetic and mathematical attitude: Utilization of previous learning, outlook, perception of value, etc.

Interest, motivation, and attitude stimulate the intellectual curiosity of the child and, as such, are important motivating forces in the study of mathematics. Each one of these is considered a mental disposition that is actively expressed on one's own from the viewpoints of "the mathematical way of thinking and expression, processing, knowledge and understanding" and

"ways of learning and problem-solving skills." These are positioned as "items that support the fundamentals and basics of content and method." The learning process of pursuing the value and significance of mathematical value is necessary in developing mental disposition.
In the following section, we will focus more particularly on fundamentals and basics related to introduce practices.

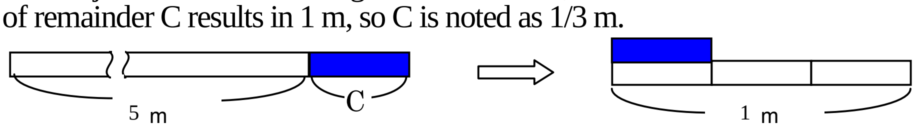
3. Formation of the mathematical way of thinking related to content: The case of thinking in terms of units

With regard to the formation of the mathematical way of thinking discussed in this paper, let us examine the way of thinking about units, from the aspect of content in particular. Let us consider the concept of units when studying the concept of numbers in the area of numbers and arithmetic (integers, decimals and fractions) and units during the numerical conversion of amounts related thereto.

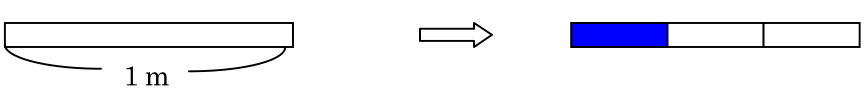
1) Formation of the concept of units during the number concept formation process

The acquisition of the number concept begins in the first year, and by looking at things from the same viewpoint, through activities in which one group is created and through one-on-one correspondence between two or more groups, pupils learn numeric representation and relations of magnitude among numbers through division into classes among these groups and through the number of class factors. Moreover, in learning about amounts and estimation—which are closely related, as will be noted later—first-year pupils form the foundation of later numerical linear algebra, as expressed by "length equivalent to x number of erasers," to take length as an example. Here, one piece is used as the unit. In their second year, pupils learn multiplication. In the case of "4 times 3," 3 is seen as a unit of addition repeated 4 times. In this case, the fact that numbers 1 through 9 are each units that can be counted is used. Such activities are the basis of pupils' understanding of the concepts of decimals and fractions over four years of instruction.
Decimals are introduced in the fourth year as numbers used in expressing the fractional part in relation to the estimation of amounts. For example, when performing a measurement using a 1-m ruler, when the measured length is "1 m plus," based on the concept of decimal notation, that "plus" fraction is processed with the concept of dividing the unit (1 m) into 10 equal parts and then using one of these 10 subdivisions as the new unit, the pupil learns numerical conversion of fractions into a number of such new units based on this.
Moreover, when measuring amounts using unit quantities—unlike in the case of decimals, where the decimal notation system is the basis for expressing fractional amounts—the introduction of new numbers and fractions comes to mind. The following two methods come to mind as such a method. Let us consider, for example, the case of 1/3.

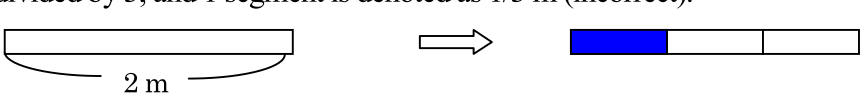
□ When an object is measured using 1 m as the unit, remainder C remains. Combining three of remainder C results in 1 m, so C is noted as 1/3 m.



□ Let us divide 1 m by 3 to obtain 1/3 m.



□ and □ are the same amount but are different ways of thinking. This difference is clearly shown in □' below.
□' 2 m is divided by 3, and 1 segment is denoted as 1/3 m (incorrect).



Both □ and □ can be called fractions that express fractional quantities in the sense that they

express an amount, but □ is an amount representation in the extension of fractions of the part-whole by equal splitting (several equivalent parts obtained by splitting the whole into equal parts) and leads easily to such a concept as that in □'. In the case of fractions of the part-whole by equal splitting, the whole is regarded as 1, and the action of splitting this whole into equal parts is emphasized, whereas when obtaining the number of split parts by dividing the unit amount of 1 m by the remainder amount and expressing the size of one such part, the remainder amount becomes the unit. In approach □, the approach of a given number of the unit fraction becomes clear. The approach of expressing 1 m with the remainder in □ is demonstrated by the Euclidean algorithm. Actually, this approach lies in obtaining the greatest common divisor c in relation to two natural numbers a and b[1].
The arithmetic action of using the remainder that remains following division as a unit that is again used for measurements is clearly different from the action of splitting the unit in □ into equal parts and using the expression as several of these parts. Moreover, it is important to consider such decimals and fractions as 0.2 and 2/7 in terms of two new units of 0.1 and 1/7, respectively. In this manner, the approach of thinking in terms of a new unit and creating such a unit plays a very important role when thinking about the four arithmetical operations for decimals and fractions. In other words, let us consider the sum of decimals and fractions as follows.

Content of "Amounts and Measurements" That Are Being Taught

In Japan, pupils learn the numerical conversion of amounts using units when being taught "amounts and measurements," broken down by school year, as follows. Such a structure aims to have pupils learn about the existence of various amounts and corresponding units as well as develop an awareness of units.

First year: Lengths (direct comparison, indirect comparison and measurements using arbitrary units)

Second year: Lengths (meaning of units and measurements and measurements using universal units (cm, mm and m))

Third year: Length (ability to make estimates and measurements using universal units (km))

Bulk and volume (concepts and measurements using universal units (l, dl and ml))

Time of day and time (duration) (concept, units (day, hour, minute and second), obtaining the time of day and time (duration))

Fourth year: Extent and area (concept, area of squares and rectangles, measurements using universal units (cm^2, m^2 and km^2) Angle size (concept and measurements using a universal unit (degree (°))

Fifth year: Quadrature formula for areas (areas of a triangle, parallelogram and circle)

Sixth year: Bulk and volume (concepts of volume, cubes, rectangular parallelepiped and measurements using universal units (cm^3, m^3)

Regarding the various types of amounts, in the measurement stage, pupils learn numeric conversion into a number of that given unit (arbitrary unit or universal unit). However, to make pupils understand the meaning of such measurements, it is important to make them practice repeatedly, using arbitrary units. Through the repeated implementation of such instruction, pupils deepen their understanding of units.

1. The Euclidean algorithm is a method of certifying the mathematical world by performing measurements with units. Assuming natural numbers a and b, with a > b, we obtain $a \div b = q_1 + r_1$ (0□r_1<b). If $r_1=0$, we obtain the greatest common denominator b. If $r_1 \neq 0$, $b \div r_1 = q_2 + r_2$ (0□r_2< r_1), and if $r_2=0$, we obtain the greatest common denominator of a, b and r_1. If $r_2 \neq 0$, the same operation is repeated, so that when $r_1 \div r_2 = q_3 + r_3$ (0□r_3< r_2) , …… , $r_n \div r_{n+1} = q_{n+2}$ results, the greatest common denominator of a and b is r_{n+1}.

2) Case: Formation of the mathematical way of thinking in courses introducing decimals

The introduction of decimals is discussed using cases (see appendix). The instructors are trainee teachers (third-year students), yet they can be called good instructors even when compared to currently active teachers. The inculcation of many different ways of thinking is sought in one class. In this class, the instructor aims to form the following types of mathematical thinking, including the concept of units.

 Understanding the concept of decimals (Concept formation: Concept of decimals)

By the time they start the course, pupils have learned the concept of integers greater than 0, the meaning of the four arithmetic operations and how to perform calculations. During the course, through the use of fluid volumes, pupils learn decimal notation using new units for amounts that cannot be expressed with integers using l as a unit. The point at which ingenuity is used in the course is when instructors have pupils perform the numerical conversion of amounts while keeping them interested by preparing the same amounts of liquid in different containers by group and using water of different colors. The pupils are made to write the amounts using cups listed in the worksheets provided by the instructor and are taught in an easy manner through activities in which they perform numerical conversion of odd amounts.

Using the fact that 1 dl, a unit the pupils have learned by then, is one tenth of 1 l, pupils are shown that this tenth is expressed as the decimal 0.1 in relation to the original 1 l unit, and they learn that 2 dl can be represented as 0.2 l.

 Forming an attitude that facilitates making estimates (Sense and thinking ability: Estimates)

One more important thing about this course is that, after providing cups containing colored water, the teacher should ask the pupils, “How much water do you think they contain?” When learning about arithmetic and mathematics, it is important to acquire a sense of quantity.

 Expressing amounts with 1 unit (Thinking mathematically: Simplification and integration)

From what they have learned up to this point, pupils have acquired the understanding that the amount obtained is “1 l and 2 dl,” but they must learn how to express this amount more simply using just one unit: l. In learning arithmetic and mathematics, pupils must understand that the simplest expression is desirable.

4. Mathematical Tools That Must Be Provided to Children

The term *mathematics class* may be misconstrued as practicing calculations. In a class where one learns and thinks for oneself, the mathematical way of thinking with regard to method and the mathematical way of thinking with regard to content are simultaneously acquired by the child. The implementation of problem-solving classes consists in allocating various problem-solving methods throughout the entire class, making pupils learn and accept each other's way of thinking, thereby teaching them the benefits of thinking. During this process, children learn how to think in relation to content and method from having to find ways to express their thoughts to each other.

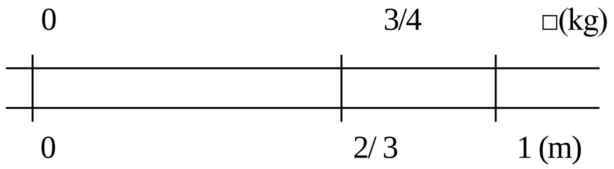
In order for children to become able to skillfully express content and method, they need to acquire the tools they will need when thinking about ways to express these things. Actually, as thoughts are exchanged, even when, for example, one feels that a tool (method of expression) or way of thinking used by a child in class is effective, there will likely be few opportunities to re-present situations so that all the children will be able to use such tools.

In the process of learning, the presentation of opportunities for the child to choose his/her own useful tools when thinking mathematically is important for forming the mathematical way of thinking. In so doing, it is also important to enable the child to use the same tools continuously and to develop these tools to create new ones. The term *problem solving* involves an emphasis on opportunities to make discoveries, but such discoveries, rather than being sudden occurrences, are for the most part achieved based on previous learning. If anything, it is important to value learning opportunities regarding ways to use tools as a part of the thinking process and ensure that children have an ample supply of such needed tools as they tackle problem solving.

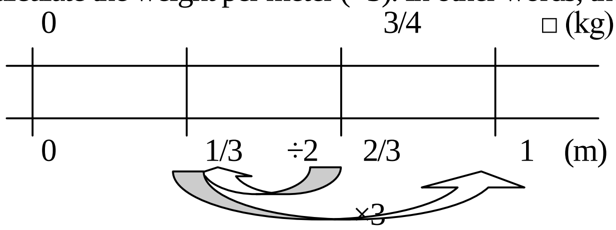
For example, in the area of numbers in arithmetic, children can use line charts or surface diagrams to show relationships between quantities and explain what they are thinking. Even though they know that these line charts and surface diagrams are effective, there are probably many children who do not understand how to use these tools. In this sense, it is important to teach them how to use line charts and surface diagrams as tools for thinking. In the following, let us take up the case of two number lines as an extension of line charts.
Example: Let us consider the following problem.

If an iron rod measures 2/3 m and weighs 3/4 kg, how many kg would it weigh if it were 1 m?

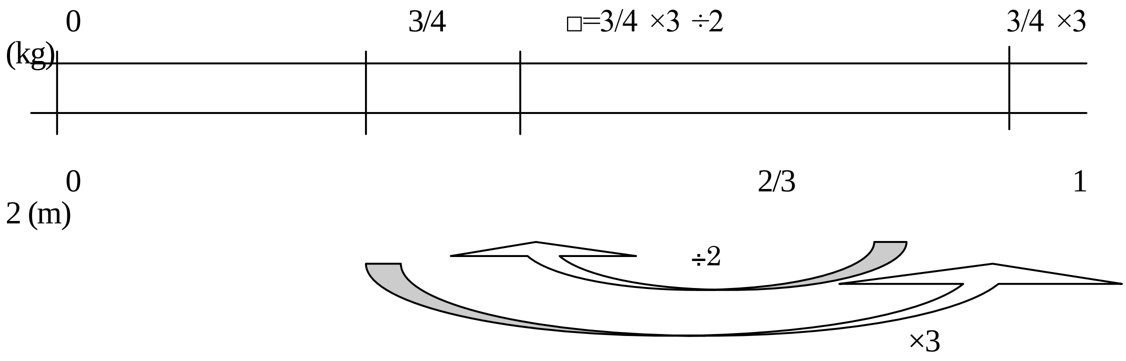
The following diagram can be used to understand that the formula for solving this problem is 3/4 ÷ 2/3 and that this is the means of solving this problem. (Because we used division to obtain the amount per meter in the integer problem, we shall use division here too.)



Using this diagram, let us think in the following way, for example, to obtain the weight of a 1-m iron rod.
□ To obtain the weight per meter, first let us obtain the weight of 1/3 m (÷2) and then calculate the weight per meter (×3). In other words, the formula is 3/4 ÷ 2/3 = 3/4 ÷ 2 × 3.



□ To obtain the weight per meter, first let us obtain the weight of 2 m (×3) and then calculate the weight per meter (÷2). In other words, 3/4 ÷ 2/3 = 3/4 × 3 ÷ 2.



In this manner, in classes that value children solving problems on their own, it is necessary that pupils learn how to use the tools that enable them to think for themselves. The ability to use the two number lines is acquired, by its very nature, through many hours of practice, and children that do not know how to use it at first gradually learn. Also, the two number lines require five or six years to master and are not something in which one can excel overnight. Textbooks are devised so that the line charts to be used by first and second year pupils are systematically and continuously addressed and are learned as an extension of line diagrams.

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Appendix

Fourth Year Elementary School Mathematics Teaching Plan

Date & Time: September 18 (Thu.), 2003; 5th period
Pupils: Fourth-Year Class 3; 15 boys, 15 girls; total: 30
Instructor: Meien Elementary School , Sapporo City(Education Trainee) Fumito Chiba

1. Unit Name: Decimals
2. Unit Aims

To teach pupils the use of decimals to express the size of fractional parts that do not reach the unit quantity and have them use decimals appropriately

- Have pupils realize that the size of fractional parts that do not reach the unit quantity is expressed with decimals and have them readily try to use decimals.
- Have pupils realize that decimals, like integers, are expressed using the decimal notation system.
- Have pupils realize that, based on the decimal system, addition and subtraction operations can be done in the same manner as for integers by calculating numbers of the same place value.
- Teach pupils how to express the size of fractional parts that do not reach the unit quantity using decimals.
- Teach pupils how to view, in a relative way, numbers expressed with decimals based on 0.1, etc.
- Teach pupils how to express decimals on a number line and read decimals displayed on a number line.
- Teach pupils how to add and subtract decimals in simple cases.
- Have pupils understand the meaning of decimals and how to represent them.
- Have pupils understand decimal addition and subtraction.

3. Regarding Course Units

Up until this point, what the pupils have learned about amounts consisted of clarifying units and learning that the number of such units that can be expressed with an integer. Here, however, pupils will learn to estimate the size of amounts smaller than units, i.e., fractions, and how to express these.

To express the size of a quantity that is smaller than a unit, one uses decimals and fractions. Here, however, the division number differs from arbitrary fractions, with the unit being divided into 10 equal parts, and decimals that can be combined in groups of 10 for a decimal scaling position (decimal system) are taken up.

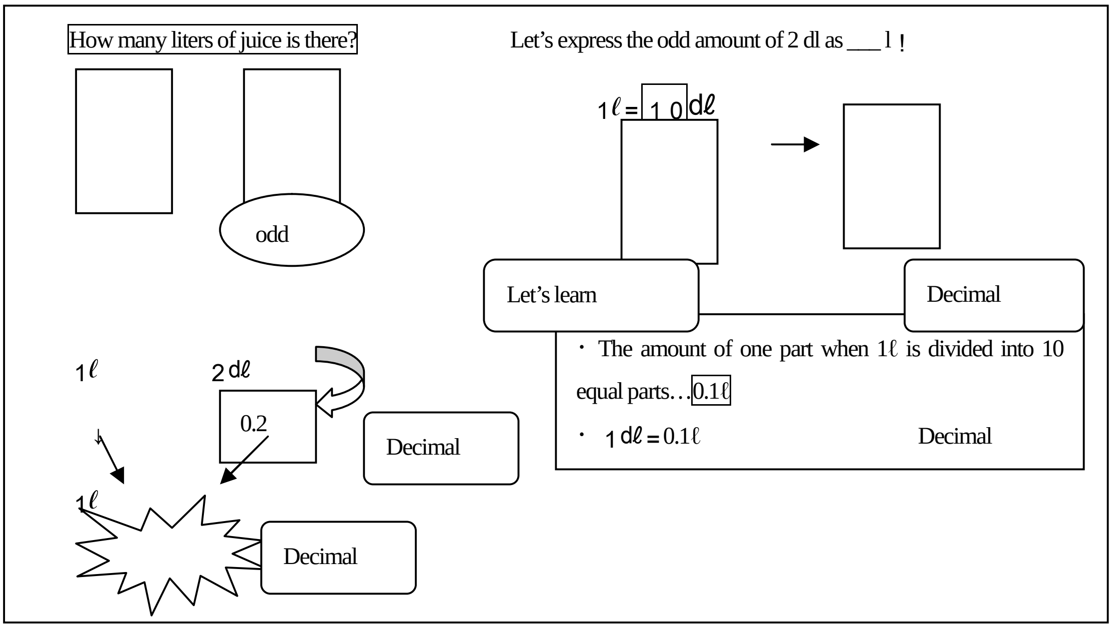
Fourth-year pupils deal with decimals through three course studies, namely, (1) the expression of fraction size, (2) the decimal system and (3) the calculation of decimal addition and subtraction on paper. This class period will introduce decimals as described in (1) and will cover amounts measured in liters. By using a new unit created by dividing 1 l, which is the unit quantity, into 10 equal parts, we will have pupils work on the size of fractions.

In this class period, by having children engage in various activities, such as using colored water to transfer an amount that is less than 1 l into a graduated container by hand while visually checking the operation, we will help them understand that odd amounts can be expressed with decimals.

4. Teaching Plan (8 Hours)

Subunit	Duration (hrs.)		Teaching Contents
1. Expression of fraction size	2	1	Using decimals to express the size of a fraction that does not reach the unit quantity
		1	The ability to express the size of fractions using decimals, even in the case of length (cm)
2. Decimal system	3	1	・ Number line display of decimals ・ Meaning of the term *one decimal place* and the decimal scaling position of decimals
		1	Relative size of decimals and structure and magnitude of numbers
		1	Addition and subtraction of decimals in simple cases
3. Addition and subtraction of decimals	2	1	Addition of decimals by hand up to one decimal place
		1	Subtraction of decimals up to one decimal place
4. Practice and review	1		Review of this class period and practice

5. Blackboard Plan



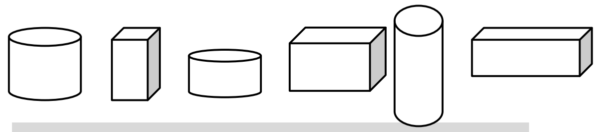
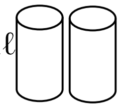
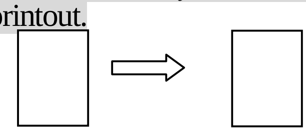
(2) Expansion of this class period (1/8 hr)

6. Lesson of This Class

(1) Course aim

Understand that decimals are used to measure amounts less than the unit quantity.

(2) Course development (1/8 of the class time)

Pupil Activities	Role of the Teacher
 ○What is the approximate amount contained?	・ Beforehand, arrange the desks to make six groups Have the children estimate the amount. ・ Explain the measurement method (1 l container without graduations) ・ Walk around the desks and check how the children are doing.
1ℓ  Odd amount ○How much is the odd amount? About 2 dℓ Less than half of 1ℓ Estimate: Total amount of juice is 1 ℓ and ○dℓ.	Note: Ensure that the children work carefully without spilling anything. ・ Define the term *odd amount*: This is an amount that is less than 1 l (unit quantity).
○What would you need to know the exact amount?	・ Have the children estimate odd amounts using dl, which they learned in their 3rd year.
Let's use a graduated container to obtain the amount. **Total juice is 1 ℓ and 2 dℓ!!** How many □ℓ of juice is there in all? Pupils realize that 2 dℓ (odd amount) can be expressed in ℓ. Let's express how many l 2 dl (odd amount) are.	・ Distribute graduated containers. ・ Explain that regardless of their different shapes, each container holds the same amount. ・ Talk about the benefit of converting 1 l and 2 dl into ___ l. ・ Distribute printouts. Have the pupils review what they have learned up to that point.
○Let's accurately write 2 dl in the left box on the printout.  Pupils realize that, similar to dl, l can be divided into 10 equal parts. Pupils learn that the amount corresponding to one of the 10 equal parts that 1 l has been divided into is 0.1 l (1 dl = 0.1 l).	・ Confirm that 1 l = 10 dl. ・ Walk around the desks and check how the children are doing. ・ For children who do not understand the concept of 10 equal parts, suggest that the height of the box is 10 cm.
Amount of odd amount: 2 dℓ = **0.2ℓ**!!	・ Explain using the figures (boxes). Do not make detailed reference to addition.

○How much is 1 l plus 0.2 l? ⇩ 1.2ℓ of juice in all Review The size of the odd amount can be expressed with decimals. Input of impressions, note regarding the next class period	

DEVELOPING MATHEMATICAL THINKING IN A PRIMARY MATHEMATICS CLASSROOM THROUGH LESSON STUDY: AN EXPLORATORY STUDY

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Abstract

This paper discusses an exploratory study that aims to develop mathematical thinking in a primary mathematics lesson. Although mathematical thinking is one of the significant components of Malaysian school mathematics curriculum, it was not explicitly implemented in many Malaysian schools due to time constraints and mathematics teachers' lack of understanding and awareness about mathematical thinking. In views of the importance of mathematical thinking, it was set up as one of the goals of a Lesson Study group existing in a Chinese primary school. Two lesson study cycles were carried out with a result of two mathematics lessons planned and observed. Five mathematics teachers participated in the study. Preliminary analysis shows that these mathematics teachers espoused that it was much easier to learn new teaching ideas such as developing mathematical thinking through Lesson Study collaboration. Initially, many teachers did not understand fully what mathematical thinking is and how to help pupils to develop this kind of thinking. After two lesson study cycles, these teachers have gained much more understanding and confidence in developing mathematics lessons that promote mathematical thinking. Nevertheless, time constraint and heavy workload remain their two main challenges to integrate any new teaching ideas and strategies.

Introduction

This paper discusses an exploratory study that attempts to develop mathematical thinking in a primary mathematics classroom through Lesson Study collaboration. A document analysis of the Malaysian primary and secondary mathematics curricula done earlier (Lim & Hwa, 2006) indicates that promoting mathematical thinking among Malaysian pupils is an intended goal but it was not explicitly spelled out in the syllabus. A literature search of local studies signify the need to have much more empirical study that focus on promoting mathematical thinking in the Malaysian classroom. Informal discussion with school mathematics teachers displayed that many mathematics teachers agreed to the importance of mathematical thinking and would like to promote mathematical thinking in their classrooms. But they are usually constrained by several issues and challenges such as (i) lack of clear understanding of mathematical thinking; (ii) lack of appropriate assessment tool that measure mathematical thinking and (iii) lack of know-how to promote mathematical thinking (Lim & Hwa, 2006).

In view of the importance of mathematical thinking and the potential of Lesson Study collaboration, an attempt was made to develop mathematical thinking as a goal of an existing Lesson Study group in a Chinese primary school. This Chinese Primary School situated in the centre of an urban area. It is a mini-size school consists of one headmistress, one male teacher, ten female teachers, and 136 pupils. There are only 6 classes with one class for each grade. The Lesson Study group of this school consisted of 8 mathematics teachers were set up since January 2006. They have gone through three lesson study cycles in the year 2006, with the aim of enhancing mathematics teachers' content knowledge and their confidence in teaching mathematics in English language (detailed report see Goh Siew Ching, 2007).

In the following section, I will first explore the teachers' understanding of mathematical thinking; follow by a brief description of the exploratory study. This study comprises of three stages: (a) an introductory workshop on mathematical thinking; (b) first lesson study cycle; and (c) second lesson study cycle. To highlight how these teachers' attempt to develop pupils' mathematical thinking, parts of the two lesson plans collaboratively designed by the Lesson Study group will be used to elaborate together with video clips of the lessons observed.

Teachers' perceptions of mathematical thinking

To elicit mathematics teachers' understanding of mathematical thinking, a brief questionnaire was given to the 6 mathematics teachers and 5 non-mathematics teachers who attended the workshop. Analysis of their response show that majority of these teachers were not sure if they were ready to promote mathematical thinking in the classroom. The main reason was "teachers are not given enough resources to promote mathematical thinking in the classroom". All except two did not answer the question, "Are Malaysian teachers promoting mathematical thinking in the classroom?" The two who answered were also not sure "because they [mathematics teachers] merely convey the knowledge of doing or solving the problems of mathematics."

Out of the 11 teachers, two of them agreed that they understand what mathematical thinking is, two disagreed while others were not sure. Consequently, only two of them agreed that they know how to promote mathematical thinking in the classroom. To these teachers, mathematical thinking refers mainly to problem solving, involve creative and logical thinking, and require skills such as reasoning, analyzing and the use of mathematical symbols. One mathematics teacher believed that she has been incorporated mathematical thinking in her daily teaching although she did not explicitly mention it in class. For her, asking a lot of "why" questions and giving pupils a variety of questions to solve are ways of promoting mathematical thinking.

An exploratory study to promote mathematical thinking

In view of the importance of mathematical thinking and the lack of proper understanding of mathematical thinking among teachers, an exploratory study was proposed to promote mathematical thinking among mathematics teachers through Lesson Study collaboration.

An introductory workshop on mathematical thinking

On March 9, 2007, all the 11 teachers attended an introductory workshop on mathematical thinking. The main aim of the workshop was to expose these teachers to the concept of mathematical thinking and to propose some possible strategies to promote mathematical thinking in the classroom. These teachers were shown a videotaped Japanese classroom lesson of a Grade 4 mathematics topic on "prime and composite number". Before showing the video, the teachers were given the same classroom activity to experience. Ten cards of different designs were arranged in a specific way. Teachers were asked to observe the order of the designs and determine what the patterns or order represent. They were then asked to identify the rules and using these rules to arrange the successive two cards. The teachers seemed to enjoy this activity and some of them were able to come out with certain kind of rules.

Later, the teachers were shown the video lesson and asked to list out the characteristics of mathematical thinking that they observed in the lesson. The following list was the outcome:

- Activity based
- Pupil centred, active pupil participation
- Justifying, reasoning, argue, debating
- Extrapolating, extend to new situations
- Generalizing, evaluating
- Decision making
- Positive attitude – willing and eager to try
- Logical thinking, creative thinking etc

Based on the list, the teachers were encouraged to plan a mathematics lesson that promotes mathematical thinking through their Lesson Study group collaboration.

Teachers were encouraged to write their reflection after the workshop. Some teachers reported in their written journals that they have been practicing some of the above characteristics of mathematical thinking in their daily class teaching. However, many of them were not aware that these were elements of mathematical thinking. They espoused that they were keen to plan out a mathematics lesson that will help to develop mathematical thinking.

First Lesson Study cycle (22 March-27 April 2007)

Five mathematics teachers participated the first Lesson Study cycle. The topic chosen was "percentage". See Appendix I for a detail lesson plan. In this cycle, the teachers met four times: 3 meetings for discussion on lesson planning and one for teaching observation followed with reflection and discussion.

Second Lesson Study cycle (13 June-16 July 2007)

In the second Lesson Study cycle, the same five mathematics teachers participated. The topic chosen was "Time" for Grade 4 class. See Appendix II for a detail lesson plan. In this cycle, the teachers met five times: 4 meetings for discussion on lesson planning and one for teaching observation followed with reflection and discussion.

General outline of the lesson

Table 1 displays the general flow or outline of the two lessons. According to the participating teachers, this is also the common format of their normal mathematics lessons. However, small group activities are seldom carried out as it is time consuming. Instead, teachers tend to explain the related mathematical concepts with examples and then give a lot of questions for pupils to practice. Nevertheless, to develop mathematical thinking, they suggested the best way is to promote through small group activity. This is because small group activity will exhibit some characteristics of mathematical thinking, such as active pupil participation, encourage pupils to present and justify their answers, and promote logical and creative thinking.

Table 1: General flow or outline of the lesson

	Lesson 1	Lesson 2
Topic (grade level)	Percentage (Grade 5)	Time (Grade 4)
Learning outcome	Convert proper fraction to percentage	Addition and conversion of time in minutes and hours
Induction set	Represent information in fraction and percentage	Link to pupils' daily life experience: favourite TV programme
Step 1	Small group activity	Small group activity 1: jigsaw puzzle
Step 2	Pupil presentation	Pupil presentation
Step 3	Practice and discussion	Small group activity 2: jigsaw puzzle
Step 4	More practice and discussion	Pupil presentation
Closure	Enrichment exercise-worksheet as homework	Enrichment exercise-worksheet as homework

Developing mathematical thinking in Lesson 1

Pupils were divided into four groups. Each group was given 3 cards, labelled as M, S and E. To stimulate the interest of the pupils, the teacher has creatively linked the cards to M for Monkey; S (Snake) and E (Elephant). Pupils were asked to write down a number between 50-100 for card M; 20-50 for card S and less than 20 for card E.

Without any prior objective of what the number will indicate, pupils simply gave a number that suited the condition. Some wrote 40 over 50 (on card S); some wrote 60 over 100 (on card E). Initially it was planned that these numbers will represent the quiz scores for each group. For example, M stands for Mathematics quiz; S for science quiz and E for English quiz. The mathematics quiz has maximum score of 100; science quiz maximum score is 50 and English quiz maximum score is 20. The pupils were then asked to write their scores in fraction form and later convert to percentage. Finally teacher asked the pupils, "Which group has the best total score to be declared as the winner of the quiz competition? What is the best way to decide?"

This was planned in such a way, so that pupils will need to rationalize [using mathematical thinking] that they have to change the score from fraction form to percentage, so that the three scores can be compared to decide the winner.

However, as reflected by the teacher, Mr L, later in the discussion after teaching observation that he has forgotten this part of the lesson plan. He forgot to ask the pupils to decide which was the best total score. Instead he asked pupils to suggest the best score for each subject.

During the teaching observation, some pupils appeared to be rather unsure about the request of the teacher. One pupil came out to give 19 over 20. But very soon he realized his error and he changed his answer to 20/20. Similarly another pupil wrote 41/100 for mathematics quiz. It was then corrected by his friend to be 100/100. These pupils' answers show that some of them understood that the best score for each subject should be 100%. Nevertheless, it was a pity that the teacher failed to grip the opportunity to encourage more mathematical thinking among pupils, by asking pupils to justify their answers.

Developing mathematical thinking in Lesson 2

Lesson 2 aims to teach the Grade 4 pupils how to add and convert two quantities of time in minutes and hours. The teacher began the lesson by asking pupils' favourite television programme and the amount of time they used to watch these programme per week. This created a cheerful discussion as all pupils were keen to share what were their favourite television programmes. To make the calculation simple, the teacher limited the number of programme to only one per day. As there was no programme on Wednesday, a total of 6 programmes were watched per week. Since each programme was shown for 30 minutes, a total of 6 x 30 minutes which equal to 180 minutes or 3 hours was the total time of watching. This was a direct and simple calculation sum for the pupils. However, to promote mathematical thinking, the teacher challenged the pupils to suggest alternative methods. One girl proposed multiple additions. She then demonstrated her method in front of the class (see Figure 1). She added 30 minutes for six times and yielded the same answer of 180 minutes in total. This is an example of mathematical thinking because pupils were encouraged to show variation in methods of solving.

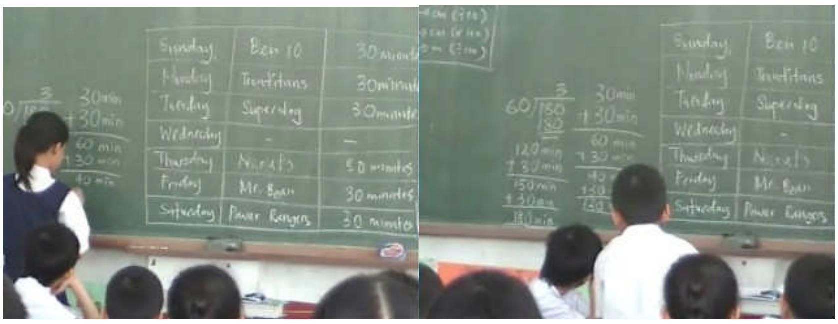

Figure 1: pupil show alternative method

In the second part of lesson 2, pupils were divided into 6 groups. Each group was given an envelope which contained two sets of question. Pupils were encouraged to discuss in group and to match every sheet of paper given to form a correct set of mathematical relationship. For example, match 3 sheets of paper as “45minutes + 50 minutes = 95 minutes”; or “35 minutes + 28 minutes = 1 hour 3 minutes”. All pupils were observed to participate actively and keenly in the given activity. Later, each group presented their solutions to the class. One pupil from each group was also asked to demonstrate their method of solving on the board.

To promote mathematical thinking, the teacher deliberately asked a lot of “why” questions to her pupils. For example, a girl pupil subtract directly 120 minutes from the total sum (see Figure 2), instead of the usual method of divide by 60 minutes. The teacher asked her to justify and the girl was able to explain that 120 minutes equals to 2 hours.

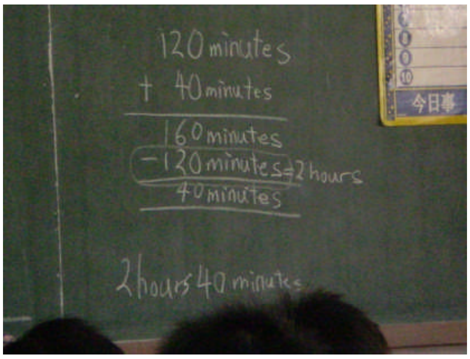

Figure 2

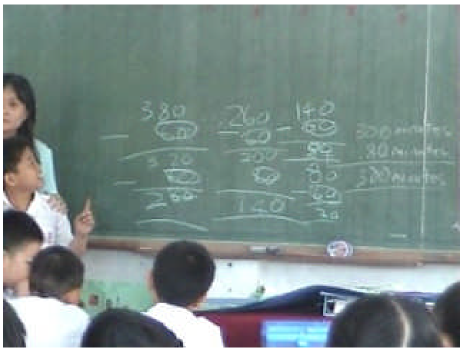

Figure 3

Similar to the first part of lesson 2, the teacher challenged frequently her pupils for alternative methods. For example, in relation to the equation: “300 minutes + 80 minutes = 6 hours 20 minutes”, both pupils displayed the same method of solving as “300 + 80 = 380” and then divide 380 minutes by 60 minutes to give the answer of 6 hours and 20 minutes. So, the teacher challenged her pupils, “Besides divide by 60, is there any other method of getting the answer?” One boy proposed, “minus!” The boy was then asked to demonstrate his method to the class. He displayed how to solve by multiple subtractions (Figure 3).

Taking this opportunity, the teacher also extended pupils’ mathematical thinking to new situations. The following dialogue demonstrates this point:

Teacher (T): Why do you circle all the 60 and 60?

Pupil (P): Because 60 minutes is one hour.

T: So you circle how many 60s here?

P: 6

T: it means how many hours?

P: 6 hours

……

The teacher also took this opportunity to ask the whole class.

T: so if take away four 60s means how many hours?

P (choral answer): 4 hours

T: how about eight 60s?

P: (choral answer) : 8 hours

Hence, it was observed that the teacher in lesson 2 was working very hard to incorporate mathematical thinking in her lesson. One also noticed that she was code-switching (using either English or Mandarin) to explain and to give instruction so as to ensure that all her pupils understand her teaching. The pupils of this class are made up of three races: Chinese, Indian and Malay. The majority of them do not understand English language very well.

Teachers' reflection

Immediately after each teaching lesson, all the five teachers and the researcher gathered to reflect and discuss. As part of the lesson study process, teachers were also encouraged to write out their reflections in their journals after every discussion and teaching observation. They were allowed to write using any language that they were comfortable with. Out of the five teachers, two of them wrote using English language, two wrote in Mandarin and one wrote in Malay language.

Teachers' Reflection on Lesson 1

The teacher who taught Lesson 1, Mr L expressed that he was rather nervous at the beginning because he was trying to recall and to follow what was planned in the lesson plan. He rated himself as 50% successfully achieved the objectives of the lesson. He was rather happy that even the 4 weakest students in his class seemed to pay attention today. He admitted that he changed what was planned in the lesson plan after the induction set.

The four teachers who observed lesson 1 expressed positive support and comments to Mr L. They contented that Mr L has clear and loud voice, very good rapport with his students, confident, patient and experienced. They also praised each other for preparing colourful power point presentation and worksheets.

One teacher, Ms S pointed that the instruction given by Mr L was rather confused. She saw many pupils did not know how to proceed, and she was rather worried at that time. Consequently, another teacher Ms M proposed that Mr L could have asked the pupils to solve based on one subject at a time and not all three subjects at the same time. Likewise, another teacher, Ms K reflected on herself that given that situation, she would quickly give examples and show to her pupils how to solve them. She was amazed that Mr L was very patient and waited patiently for his pupils to explore and to find out the answers by themselves.

When asked if they have incorporated mathematical thinking in that lesson, they all agreed that they have attained to a small extent. For example, when the teacher asked, “what will be the percentage if there are only 10 questions?” This kind of question encouraged pupils to extend their understanding to solve another kind of questions or there is variation of questions. However, due to time constraint and the pupils’ ability, they found it difficult to integrate much mathematical thinking in the lesson.

Nevertheless, when challenged to suggest other possible ways of integrating mathematical thinking in this lesson, they suggested the best way as asking a lot of “why” questions. For examples:

“why must be divided by 100 to get the percentage?”

“why converting from fraction to percentage, we use multiplication? But converting from percentage to fraction, we use division?”

“Why do we need to score full mark to be the winner?”

Another suggestion was encouraging pupils to give alternative methods of solving.

Teachers’ Reflection on Lesson 2

Teacher who taught lesson 2 was Ms M. On reflection, she acknowledged that she did not follow the lesson plan strictly. She did not manage to cover all parts of the lesson because she believes that, “if pupils could not understand, there is not point to go on.” Due to short of time, she changed the last part of the lesson to ask pupils to continue the following day. For her, today’s lesson was not of any special but as what she normally did in class. However, her colleagues who were observing Lesson 2 felt that the class atmosphere was very lively and pupils seemed to enjoy the activity. All teachers were amazed with the number of TV programme and the familiarity of the pupils about these programmes.

Mr L observed that some pupils were able to explain the alternative methods that they suggested, this shows that they were thinking mathematically. He found some pupils were arguing among themselves when they were doing the matching activity. Some pupils used trial and error, some started to write down and calculate. Most pupils seemed engaged and enjoyed themselves. Ms K and Ms S echoed that they were a bit worried that the pupils could complete the matching activity successfully. This was because they have attempted to solve the problems while preparing the activity. It took them quite some times to find a match for one of the questions. They were very happy to see that all pupils could find the answers correctly.

Ms C gave some suggestions for improving the teaching such as pasting the questions on the board so that every pupil can refer to the question. She also suggested that besides the multiple additions (30 + 30 + …) and multiplication (30x6), another way is grouping of 30 + 30 become 1 hour, so 3 groups of 30+30 become 3 hours.

All the teachers agreed that although Lesson 2 appeared simple and easy, teacher Ms M has managed to incorporate mathematical thinking in the lesson. The teacher has asked a lot of “why” questions and has always encouraged pupils to suggest alternative methods of solving. She also encouraged pupils to present their solutions in front of the class.

Teachers' Reflection on Lesson Study

All teachers agreed that participating in lesson study gained them a lot of new ideas and new experiences. They felt better collegial collaboration with their colleagues. However, in spite of the benefits, they felt lesson study was a challenging task. They lamented that each lesson plan using the lesson study cycle required at least 3 to 4 weeks to be completed. In view of the present school system, they were overloaded with tons of paper works besides teaching load. They were over-stressed and rather reluctant to continue lesson study process. These grouses were also reflected in their journal writing. This shows that "time" remains the biggest challenge to the sustainability of lesson study process.

Teachers' Reflection on mathematical thinking

After the two cycles of lesson study, I discussed with the teachers about their understanding and importance of mathematical thinking. The school principal also joined us for the teaching observation of Lesson 2 and the reflection and discussion after that.

Ms C commented that she used to promote mathematical thinking in her normal class, such as variation in difficulty level (from easy to difficult), variation in types of question and variation in methods. However, she was not aware about the term, mathematical thinking. She believes that it is pertinent to encourage pupils to think mathematically. Mathematical thinking should be an important part of mathematics learning.

Mr L supported Ms C's comments about the importance of mathematical thinking. He remarked that mathematics lessons that involve activities that promote mathematical thinking appear more lively and enjoyable. By encouraging pupils to use various kinds of methods will make them more flexible in thinking. This might enhance their adaptability to daily life and future career. All the other teachers also agreed that the normal mathematics lessons are usually very boring and inflexible [死扳]. Pupils are usually asked to follow exactly what the teacher taught. Hence, mathematics lessons should include activities that promote mathematical thinking. The school principal especially agreed that it will be ideal if every mathematics lesson can help to develop pupils' mathematical thinking.

However, time remains the biggest challenge for these teachers. They lamented there were too much workload and documents that they have to prepare daily. Mathematics lessons that promote mathematical thinking usually take time to prepare and to engage pupils to participate. In addition, with the present school system that emphasis on examination, teachers and pupils are forced rushing to finish the syllabus, and to ensure pupils are prepared for examinations. Hence, it is too challenging and stressful to incorporate mathematical thinking in every mathematics lesson unless there is reform in the present school system, examination culture and emphasis of mathematical thinking.

Conclusion

This paper reported an exploratory study that aimed to promote mathematical thinking among pupils of a Chinese primary school in Malaysia. Even though all the five mathematics teachers participated in this study were familiar with lesson study process, they were not clear how to help pupils develop mathematical thinking. For these teachers, several ways of promoting mathematical thinking are (i) ask a lot of "why" questions; (ii) encourage alternative methods of solving; (iii) variation in types of question, so that pupils are encouraged to extend their knowledge to apply to new areas.

These teachers agreed to the importance of mathematical thinking and were keen to promote it. However, they remain sceptical about the practicality and feasibility of this project. They lamented the biggest challenge is time factor. They consider a mathematics lesson that promotes mathematical thinking to be always time consuming and effort driven. Nowadays all teachers are overloaded with both teaching and non-teaching duties. They are always stressed by the school authority and pupils' parents to complete syllabus in time and to ensure their pupils excel in public examinations.

In brief, the experience of this exploratory study implies that it remains a big challenge to promote mathematical thinking in Malaysian schools. Several hindrance are (i) school culture; (ii) teachers' attitude and commitment; (iii) teachers' workload; (iv) exam-oriented culture and (v) assessment system. Unless there are efforts to reduce these hindrance, or else the road to promote mathematical thinking in Malaysian mathematics classroom seems to be still far-fetched.

Acknowledgement

This study was made possible with the cooperation, sacrifice in time and effort of Ms Goh Siew Ching and her school principal and colleagues, as well as the pupils of the two classes. Special thanks also to my two postgraduate students, Hwa Tee Yong and Gan We Ling in helping to video tape the lessons.

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Appendix I: **Lesson Plan 1**

Subject	:	Mathematics
Year	:	Year 5
Learning area	:	Percentage
Sub-topic	:	(a) Convert proper fractions to percentages. (b) Convert percentages to fractions.
Duration	:	60 minutes
Resources	:	Blackboard, manila cards, marker pens, cardboards, hundred square paper, LCD, laptop
Key words	:	Percentage, symbol, percent, hundredths, hundreds squares, parts, convert, fraction, denominator, numerator, equivalent, simplest form.
Learning Objective	:	Pupils will be able to understand and use percentage.
Learning Outcomes	:	Pupils will be able to 1. Convert proper fractions to percentages. 2. Convert percentages to fractions in its simplest form.
Previous knowledge	:	Pupils have already learnt the name and the symbol for percentage.
Values	:	Self-reliance, logical thinking, mathematical thinking, cooperative, bravery, gratitude, careful, helpful.

Appendix II: **Lesson Plan 2**

Subject	:	Mathematics
Year	:	Year 4
Learning area	:	Unit 5 Time
Sub-topic	:	Basic operations involving with time: Add minutes with answers in hours and minutes
Duration	:	60 minutes
Resources	:	Blackboard, manila cards, marker pens, cardboards

Key words : Convert, relationship, involving

Learning Objective : Pupils will be able to do the basic operation involving time.

Learning Outcomes : Pupils will be able to

3. Add minutes with answers in hours and minutes.
4. Convert units of time involving hours and minutes.

Previous knowledge: Pupils have already learnt time in hours and minutes and converting units of time involving hours and minutes.

Values : Logical thinking, mathematical thinking, cooperative, bravery, honesty, careful, helpful.

Step	Content	Activities		Remarks
		Teacher	Student	
Set induction (± 1 minutes)	Asking questions related to their daily life.	T: What is your favourite TV programme?	Various answers will be given by the pupils .	Pupils listen and respond.
Development 1 (± 9min)	Variation of questions .	Teacher asks the following questions and led the pupils to answer. T: So, how many minutes you spend to watch your favourite programmes on Monday? T: Tuesday? Wednesday?..... T: How much time do you spend on watching TV programs in a week? Teacher will draw a table concerned on	Various answers will be given by the pupils Pupils find the duration of the time spent for TV programmes on each day and the total time spend in a week.	Pupils listen and respond.

		the blackboard and asks the pupils to find the duration of the time spent on TV programmes each day and the total of time spent in a week. Teacher also remind the pupils the moral value behind it, i.e. don't spend too much time on TV programmes, but instead have to choose the good programmes.		
Development 2 ($\pm$ 46min)	Jigsaw puzzle: two sets of questions i) Easier Questions: Purple, pink, Green cards. (20 minutes) ii) Difficult Questions : Blue, yellow, orange cards (20 minutes)	1st round: Easier 1. Divide the pupils into 6 groups. Each group consists of 4 or 5 pupils. 2. One representative of each group comes forward to get an envelope. 3. Inside each envelope, there are 2 pairs of questions. 4. Every pupil in each group think, discuss and to match the correct pairs of questions and answers so as to finish the task. 5. Teacher will then ask the pupils to come out to explain how they get the answers and discusses with the pupils.	Pupils cooperate to find the correct pairs of questions and answer and then paste the answer on a manila card in each group. Pupils present their 'works' on the blackboard. Pupils come out to explain how they get their answers.	Pupils discuss and solve the problems.

		2nd round: Difficult 1. Repeat steps 1to 5 as in 1st round.		
Closure (± 4 min)	Conclusion Enrichment	Teacher concludes the lesson. Every pupil will be given a copy of worksheet as homework for enrichment.	Pupils listen and solve the problems.	Worksheets

DEVELOPING MATHEMATICAL THINKING THROUGH LESSON STUDY: INITIAL EFFORTS AND RESULTS

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This paper describes how through lesson study two teachers were made to experience mathematical thinking so that they in turn could create opportunities for their pupils to experience it. In developing the lesson intended for this, they realized the need to deviate from many of their long-held unquestioned practices. This meant trying out what they have not done before in their teaching. Despite some lapses due to their adjusting to the changes in their practices, their actual teaching of the lesson showed that their initial efforts could engage pupils in mathematical thinking. And so, this was a good start.

ANALYZING EXISTING CONDITIONS

Identifying the Usual Practices

The usual components of an elementary mathematics lesson are: drill, review, presentation, developmental activity, fixing skills, generalization, application, and evaluation. In the presentation, a word problem serves as a source of the numbers that are computed by the whole class through the guidance of the teacher during the developmental activity. After the pupils have read the word problem that is expressed in English, it is a standard procedure that they are made to answer guide questions to help them understand and analyse it. Filipino is the native language but English is used to teach mathematics. The guide questions are: (1) What is asked (A)? (2) What is given (G)? (3) What is the word clue/operation to use (O)? (4) What is the number sentence (N)? (6) What is the answer (A)? AGONA implicitly shows how a word problem should be analysed. (Department of Education, 2002).

Determining the Need for a Research Lesson

Last year, the teachers identified that their lesson study goal is to increase pupils' motivation in learning mathematics and to improve their comprehension and analysis of word problems. For this year, they wanted to develop a lesson on solving problems involving subtraction of whole numbers with regrouping because it is a very difficult topic for many pupils. Difficulty is on understanding a foreign language and also on the process of regrouping. To help pupils understand and analyse word problems, teachers code switch and allow the pupils to talk in Filipino or code switch. They teach pupils to look for clue words, words that are associated with a particular operation, so that they will know what operation to use. Examples are deduct, reduce, less, and take away which are associated with subtraction. Relative to forming a number sentence involving subtraction, teachers observe that pupils tend to write the number mentioned first in the word problem as the minuend even if this is actually the subtrahend or the difference. Apparently, the pupils do not understand the word problem. Moreover, they note that in finding the difference of two whole numbers with several digits, there are pupils who always subtract a smaller digit from a bigger digit with the same place value disregarding whether the smaller digit is in the minuend or subtrahend. In effect, they do not use regrouping. The lesson study group has to develop a research lesson that takes into account all these conditions and at the same time develop mathematical thinking among pupils.

Examining the Usual Practices

Since grade 1, pupils have been made to do AGONA as a whole class and it takes time. The teachers were asked to consider other ways to help pupils understand and analyse word problems. These include making them relate what they understand about a it using their own words or asking them what the answer to it is and to explain how they got this. These require pupils to make sense of the word problem. The teachers were also challenged to make their pupils solve the word problem on their own without their guidance. They predicted that many would not be able to do it but a few would. But they were willing to try. They were reminded of the importance of enabling pupils to think on their own.

The teachers were also asked why they teach their pupils to rely on clue words to determine the operation to use in a word problem. It was explained to them that this might not really help pupils' comprehension for they might just look for the clue-words and no longer try to understand the word problem. They were given an example where there was no clue word. (There are 3 500 balls. There are 1 750 balls that are not in the boxes. How many balls are in the boxes?) They were also given a counterexample where the word that is thought to be associated with a particular operation is not so. (The grade 4 classes collected 2 478 stamps. The grade 3 classes collected 1 543 stamps. How many more stamps did the grade 4 classes collected than the grade 3 classes?). Lastly, they were made to realize that word clues are often limited only to the "take away" interpretation of subtraction and disregards its other meanings, namely: additive, comparative, partitive, and incremental (Troutman & Lichtenberg, 1991). Since the teachers were not aware of these other interpretations, they were asked to write word problems on them.

DEVELOPING A LESSON THAT ELICITS MATHEMATICAL THINKING

Using a Framework on Mathematical Thinking

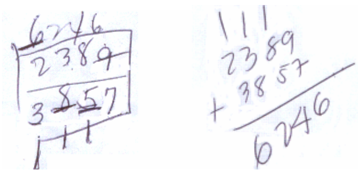
In developing the lesson, the following were considered: teachers must engage in mathematical thinking so that they can elicit this also in pupils; teach mathematics through problem solving to integrate mathematical thinking in the learning of content and enhance pupils' reasoning and communicating skills; encourage pupils to think through questioning; ask pupils to discuss their ideas and formulate their own problems; use non-routine tasks such as open-ended problems to develop mathematical thinking in routine tasks; enable pupils to use their previous knowledge and skills to unfamiliar contexts; connect concepts and procedures; anticipate various pupils' responses; and develop mathematical attitudes like persistence in solving problems and verifying results (APEC Organizing Committee 2006) .

Engaging Teachers in Mathematical thinking

Considering that the procedure in subtracting whole numbers with regrouping had been introduced since grade 3 only with fewer digits, and assuming that pupils could understand English, then this topic would not present anything new from what had already been taken previously. And where would mathematical thinking naturally fit in? So the teachers were probed on what other lessons they had taught so far about subtraction. They had given exercises on finding any of the following, given the other two: difference, subtrahend, or minuend. Building on this, the teachers were then asked in which word problem the pupils would have more opportunity to think - finding one of the following given the other two: difference, subtrahend, or minuend or finding the missing digits in the minuend, subtrahend, and difference which are all given. They were then made to answer the following:

Mr. Jose saves money for his house repair. The repair costs P _246. He has already saved P238_. So he still needs to save P3_ _7. How much does the house repair cost? How much has Mr. Jose saved already? How much more does he need to save?

Shown below is the work of one teacher who consistently used addition to find the missing digits.



The other teacher used a combination of addition and subtraction. Both of them clearly explained their work. They used the relationship of addition and subtraction, the concept of place value, and the process of regrouping. They appreciated solving this word problem involving missing digits and their using different ways to find them. They remarked that this was the type of word problem that could make their pupils think more. Thus they were asked to make similar word problems using the different meanings of subtraction. It was intended that later they would be given an example of an open-ended problem involving numbers with missing digits.

So the teachers made word problems. They were systematic in their explorations. One placed a blank in each column of the number sentence from right to left and in the subtrahend, then minuend, and the difference. The other placed a blank in each column of the number sentence except in the tens place where he placed a blank each in the minuend and the difference. The blanks were distributed in the minuend, subtrahend, and difference. To his surprise, he got many different possible answers. He asked that if this was the case, then what would the correct answer be. His question provided the opportunity to introduce to the teachers what open-ended problems are. He was told that the problem he made was one example. He said that in previous seminars that he had attended, he had already encountered this term. But it was only then that he understood what it meant. Since the teachers engaged in mathematical thinking, they were better prepared to engage their pupils in this experience, too.

While developing the lesson, the teachers realized that the pupils would need time to solve problems. So they decided not to have a drill and review and to have the problems right away. The problems that the teachers made are shown in the lesson plan. Together, the group thought of possible ways that the pupils might solve them.

Preparing the Task

The task consisted of two problems. Problem 1 was supposed to familiarize pupils with subtracting numbers with missing digits. Problems 1 and 2 were similar because their numbers both had missing digits, and they could be solved in different ways. They were different in that the number sentence in Problem 2 had a column with two missing digits and so was open-ended while in Problem 1 each column of the number sentence had only one missing digit and so was not open-ended. Problem 2 had many different correct answers while Problem 1 had only one. Problem 2 used the comparative meaning of subtraction while Problem 1 used the partitive meaning. In Problem 2 a smaller number came first before a bigger number while in Problem 1, a bigger number came first before a smaller number.

In developing the lesson, it was mentioned that a pattern can be observed as one systematically replaced the two blanks with different digits in the tens column of the number sentence in Problem 2. However, time was not enough to go deeply into this. In the lesson itself, it would be enough that pupils would realize that they could substitute different digits and get different correct answers. This would be the first time that they would be solving a problem like this. The teacher themselves would have to investigate the following: If one systematically substituted the digits 0 to 9 in the minuend, then the digits in the difference would be the same. Why? The higher the digit that was substituted in the minuend, the higher would be the digit in the difference. Why? The higher the digit that was substituted in the difference, the higher would be the digit in the minuend. Why? Would these results still be true if the given digit was no longer 9? Why? Why was it that it was only in the tens place of the minuend and difference that the digits differed? Why were the digits the same for all the other place values? What would be the answers if instead of the digits missing on the minuend and the difference they were missing on the minuend and the subtrahend or the subtrahend and the difference? What if instead of two digits on the same column, two consecutive digits on the same row were missing? These questions show that the task is mathematically rich.

ELICITING MATHEMATICAL THINKING IN PUPILS

Problem 1

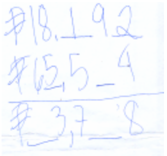
By asking a question that relates to their daily life experience, the teacher got the pupils' interest in the lesson. Being interested can facilitate thinking. By making the pupils read the situation, in particular the numbers that have missing digits, they were made to think because this was the first time that they had something like this. They needed to relate their understanding of place value in order to read correctly each number. Some of them were able to. This was an instance of connecting their existing knowledge to a new context. By making them write the numbers, he focused their attention to the numbers' having missing digits and tried to make them realize the need to find these digits first to answer the questions. Focusing is an important thinking skill.

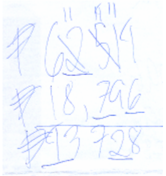
The pupils were not asked to answer AGONA. They were expected to understand what they read and figure out on their own how to answer the questions. Understanding the situation and analysing the relationships implied there without the teacher's guidance required thinking. This was not what they were used to do. Moreover, there were no clue words mentioned on which they could rely. When the teacher asked if they could form a number sentence for the situation, they at first said no. When he raised the question again, they hesitated to answer until he said that there was a problem expressed in what they read. This was the first time that they encountered a problem presented in this way. The number sentences that they had formed before were for those problems that they were familiar with. In what they had now, every number needed was already given but had missing digits. What they had done previously involved two numbers without missing digits and they had to find a third number by applying a given operation on them. So they must be thinking how they would connect what they knew with this new one. If they truly understood what they read, then they could write a correct number sentence that is, represent the relationship symbolically, which is an important thinking skill. In particular, as the video shows, the pupil who was called to write the number sentence was unsure of what he was doing. But with the teacher's coaching, he was assured that what he was doing was right. This action of the teacher was important. He did not give the number sentence himself but gestured approval of the work of a doubting pupil.

After one correct number sentence was given, the teacher asked the class to work in pairs and discuss. Everybody worked and most of those who had seatmates discussed with each other. This discussion provided opportunity for the pupils to think together in determining the digit that should go into each blank. Then, he asked some pupils to explain their work. Explaining one's work required thinking. One had to be able to clearly, completely, and correctly describe one's work in order for others to understand and be convinced that the work was correct. Some explanations were more conceptual than the others. The pupils gave reasons for how they obtained the digits using the inverse relationship of addition and subtraction, the concept of place value, and the process of regrouping. It looked like this problem was easy for most students. Based on the numbers they obtained with completed digits, the teacher asked if they could give another number sentence. He also remarked that if they knew any two numbers, then they could get the third number. The pupils added the subtrahend and the difference and got the minuend. Indirectly, he made them verify if the numbers they got were correct. Implicitly, he also emphasized that addition and subtraction are inverse operations and when two numbers are known, a third number can be known given an operation. Later, the pupils would meet the latter relationship again in the next problem. He also established that now that the numbers did not have missing digits anymore, what they had done before with such numbers could be applied with these ones, too. Recognizing and making all these connections are important mathematical thinking skills. Forming the habit of always checking one's answers is very important. He commented that all their answers were correct although they had different solutions referring to the ways that they used to find the missing digits.

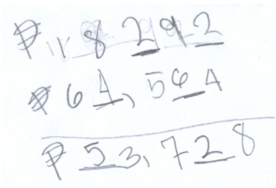
Problem 2

Basically, the teacher's approach was the same as that in the first problem except that he did not ask the pupils to write each number anymore. And so his actions that elicited thinking in them earlier did the same here because in this problem, the numbers were larger and a smaller number was mentioned first before a larger number. When asked to write a number sentence, a pupil wrote this smaller number first then the larger number. Perhaps, because he was not used to being left on his own to understand and analyse a word problem, he did not fully understand it. No one called the attention of the teacher about this nor questioned him or the pupil who wrote it why this was so. Only one pupil realized the error. As shown below, he wrote the bigger number first followed by the smaller number on his paper even before the teacher called their attention.

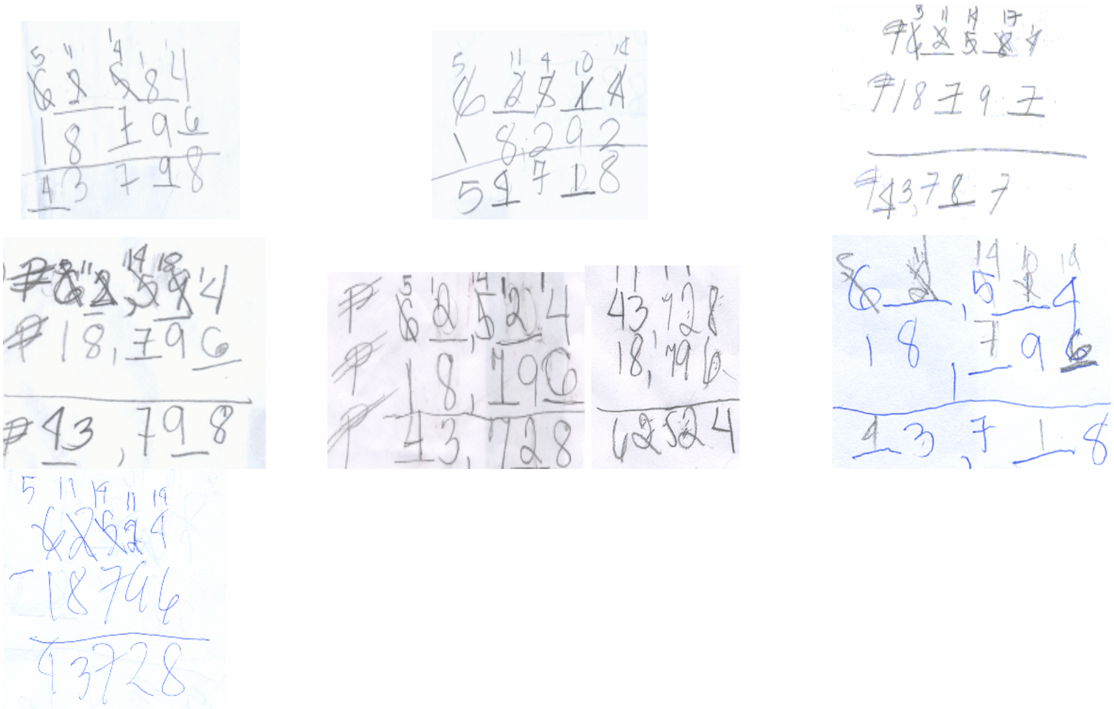




This incident provided an opportunity for pupils to really think. Every one tried to make sense of the number sentence and how to carry out subtraction to get the missing digits. According to an observer, a pupil adjusted the alignment of digits so that 6 is under 8 and so 18 was bigger than 6. Then she affixed 0 at the units-digit of the number above so as to align all the digits correspondingly with those in the number below. Others subtracted the digits in the number below from their corresponding digits in the number above from right to left. When they reached the ten thousands place, they reversed the direction from top to bottom as shown below.



When the teacher realized the mistake, he called the attention of the class by asking which number was bigger. He did not say that the number sentence was incorrect. He wanted the class itself to realize this. And the boy who had corrected the mistake earlier said that the second number should be on top because it was bigger and the number above should be the second number because it was smaller. But apparently, the teacher wanted a conceptual reasoning like this: All the numbers have the same number of digits and the second number has the highest ten thousands digit so it should be written first for the subtraction sentence to be correct. So his question was intended to make pupils think why the number sentence was incorrect. They simply responded “6.” A girl wrote a different number sentence. She must have reasoned that the largest number should be the sum of the two smaller numbers so a number sentence could be one that involved addition. However, he was trying to correct the subtraction sentence, so he did not pursue her answer. Nevertheless, this shows that pupils had different number sentences in mind and that they understood relationships. This indicated that they were thinking. Finally, the number sentence that he expected was written on the board. And the pupils again worked on their seats. Shown below are the works of several pupils. Although some of their answers were incorrect, still these showed that they had an idea that they could substitute different digits to the blanks in the tens place of the minuend or difference. Most of these pupils checked their answers using addition.



Then, the pupils were asked to show and explain their work. It was observed that a pair of pupils was not following the class discussion because they were so preoccupied with finding the missing digits and checking their work. It was not clear in the explanations of those who were called how they determined the missing digit in the tens place of either the minuend or difference, although this must have required thinking from them. The teacher asked if these answers were correct by making them form another number sentence based on their answers. He commented that all their work were correct although their solutions were different.

Presented on the board, were two different correct answers. The teacher called the attention of the class to this by saying: "Look at this number and this number. Is there a difference?" This was to show that for this problem, they had more than one correct answer. In ending the lesson, he asked how many digits were missing in the number sentence in Problem 1. What he actually referred to was the number of missing digits in the tens column. However, this was not clear to the pupils. He also asked for the number of missing digits in the tens place of the number sentence in Problem 2. Although he had no time to elaborate because the class was over, the teacher was attempting to help the pupils recognize a relationship. There was only one correct answer in Problem 1 because there was only one missing digit in the tens column of the number sentence. This meant that two digits were known, so the third digit could be determined using the given operation. In Problem 2, there was more than one correct answer because there were two missing digits in the tens column of the number sentence. This meant that only one digit was known so another digit could be chosen, to get a third number. And there could be more than one choice.

REFLECTING ON NEEDED IMPROVEMENTS

In doing some things for the first time to engage his pupils in mathematical thinking, the teacher showed some good qualities. With commitment, he actively participated in developing an interesting and challenging lesson. In teaching, as much as possible he drew out from the pupils the correct responses by asking them questions. He also gave them enough time for problem solving. But because he was still adjusting, there are still aspects in his teaching that could be improved.

Processing of Pupils' Responses

Proper processing of pupils' responses is very important. The teacher needed to pay more attention to the correctness of their responses given orally or in writing on the board. He missed instances that could have been used to deepen their understanding of mathematics and made them think more deeply. An instance was the misreading of the numbers with missing digits. If he had caught the error and asked them to read correctly, then the number sentence for Problem 2 could have been written correctly. But given what had happened, he should have asked them if they noticed the mistake and that if such happens again next time, then they should ask him. It is important for pupils to have a questioning attitude. Another occasion was the writing on the board of incorrect answers such as the incorrect number sentence for Problem 2. If the error was identified at once, then more time could have been used for exploration in Problem 2 so pupils could possibly observe patterns and make conjectures that are very important in mathematical thinking (Stacey 2006). Another instance also was a boy's giving of the correct answer that the bigger number should be written above the smaller number. He could have probed for the reason for it and used it to lead the class to his expected reasoning. And still another instance was a girl's giving of a number sentence that he did not expect.

He expected to get:

```
  6_ 5_4
- 18 _9_
---------
 _ 3 7_8
```

But the girl wrote:

```
  18_ 9_
+ _37_8
---------
  6_5_4
```

This could have been an opportunity to show that even if different number sentences were formed, the different correct answers could still be obtained.

Asking pupils to explain their work is another aspect that needs improvement. The pupils must have done some reasoning to find each missing digit in a number. But when they were asked to explain their work, some simply described how they used subtraction with regrouping in their number sentence. They did not give the reasons on how they found the missing digits in a similar way that the teacher had explained when he worked on the problem in lesson planning. Through questioning, he could have enabled them to explain similarly. This could provide a conceptual meaning to procedures that is important in mathematical thinking.

Giving feedback on the correctness of a pupil's answer by the teacher's probing into his/her response or asking the class to comment on his/her work is also important. Without this, pupils would not know how to move on. This happened to a pupil who was called twice but because he did not know if his first answer was correct, he spent much time checking his second answer that depended on the first. The teacher should also have noted if pupils were careful to use the correct mathematical symbols for their work to be meaningful.

Additionally, board work has to be improved. Writing should be organized and should not be erased to provide a sequential record of what transpired in the lesson and help summarize important points (Wang-Iverson & Yoshida, 2005). The teacher could have guided the pupils in organizing their board work.

Helping Pupils Make Connections

There were several opportunities to emphasize relationships so that pupils could view the lesson coherently. After the missing digits in every number had been determined, the pupils could have been asked to interpret what those numbers were by relating them to the original problem situation. As it was, the pupils simply looked for the missing digits in every number. If the teacher had provided for this, then those who did not put the peso sign would have realized that their answers were meaningless.

Connections could have been made also by focusing on the features of the two problems through asking pupils to compare their similarities and differences. That the second problem could have many different correct answers that could be obtained in many different ways could have stood out if the teacher had asked the pupils to tell what they had observed about the two number sentences for the two problems. He could have done this when the pupils seemed to consider that only one digit could be correctly placed in the blank for the tens digit of the minuend in the second problem.

It is also important for the teacher to involve the whole class in analysing pupils' work. He could have asked them to compare their work and reasoning with those presented on the board and explained in class and to give comments about them.

Making the Most out of Pupils' Work

By consciously spotting the different answers of the pupils while they were working and later calling them to show and explain their work on the board, the teacher could have gathered more different answers and solutions that represented different ways of thinking. From these, he could have asked what they observed about the answers and why they are different but still correct. However, he only called those who raised their hands and so it seemed that the same pupils were answering. He should also have seen if the pupils were really discussing while they were solving the problems.

CONCLUSIONS

With commitment and courage and having engaged in mathematical thinking, the teachers developed and taught a lesson that they had not taught before and in ways that they have done for the first time. They were certainly trying to adjust. They had done the best that they could to introduce problems that would develop pupils' mathematical thinking. Despite certain teaching aspects that need improvements, pupils' responses indicated that they were capable of engaging in mathematical thinking. When the teachers become more at home with their changed practices and engage more in mathematical thinking, possibly the pupils could engage better in mathematical thinking.

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BRIDGES AND OBSTACLES: THE USE OF LESSON STUDY TO IDENTIFY FACTORS THAT ENCOURAGE OR DISCOURAGE MATHEMATICAL THINKING AMONGST PRIMARY SCHOOL STUDENTS

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The first part of the paper describes a study conducted to explore the use of lesson study as a professional development tool. The specific aims of the study and the main stages of the study are described. The second part describes how lesson study helped teachers understand the factors that encourage or discourage mathematical thinking during a lesson. The third part briefly discusses the role of lesson study in enhancing teachers' pedagogical knowledge. The final part outlines a research agenda that employs lesson study to help teachers develop approaches to cultivate mathematical thinking amongst students.

INTRODUCTION

The Singapore mathematics curriculum which focuses on mathematical problem solving was introduced in 1992 and was revised in 2001 and, again, in 2007. Increasingly, the shift with each revision of the curriculum is less emphasis on computational, procedural skills and more emphasis on mathematical thinking. Mathematical thinking is integral in the process of problem solving.

It is, thus, important for teachers to understand the idea of mathematical thinking and how to cultivate it amongst students. However, teachers need to re-examine their own mathematical thinking and their perception of what mathematical thinking is.

Lesson study provides a concrete image and specific situations of mathematical thinking amongst students as they unfold in a classroom. The research lessons provide opportunities to capture the complexities in understanding what mathematical thinking is and the pedagogy associated with its development, that otherwise may not be captured.

THE STUDY

A group of eight teachers in a primary school in Singapore was involved in a six-week lesson study cycle. The aim of the study was to explore the use of lesson study as a professional development tool. In particular, the study reported in this paper focused on two goals. One goal was to enhance the teachers' pedagogy with respect to cultivating mathematical thinking. The other goal was to enhance the teachers' own mathematical thinking and their understanding of mathematical thinking.

In the first session, the teachers were familiarized with the ideas of visualization and generalization as possible aspects of mathematical thinking (Yeap, 2006). The teachers then used a topic (angles) that they were going to teach in the coming weeks to anchor their discussion. The teachers studied the textbooks, workbooks, teachers' guides and other resources that were available including manipulative materials. The discussion culminated in the teachers identifying ideas in the topic of angles that would be a challenge or otherwise for the primary four (grade four) students. The research theme for the research lesson was decided to be helping student construct a visual representation of angles with a focus on a representation that was thought to be challenging for the students. It was thought that students find it difficult to form a visual representation of an unknown angle a when $a + b$ is known (Figure 1).

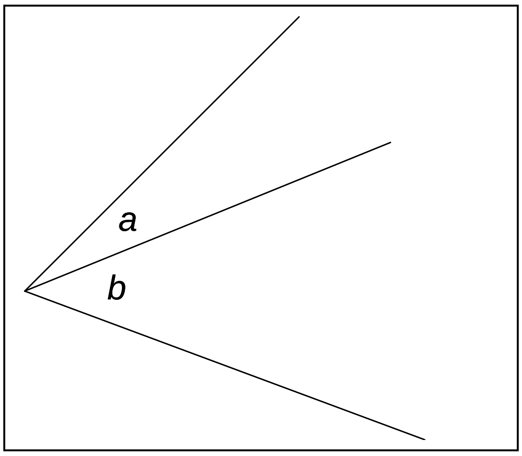

Figure 1: While students are able to tell the angle $a + b$ when angles a and b are given, they find it more challenging to tell an unknown angle when $a + b$ and either a or b are given.

In the second session, the teachers designed the lesson, wrote the lesson plan and made the necessary preparation for the research lesson. They worked out the solutions of the problems they planned to pose to the students and the anticipated students' responses.

As the focus of the lesson study was to develop a possible approach to help students construct a visual representation of angles with a particular emphasis on representations that are considered challenging to the students, the teachers decided to employ the use of concrete materials. Cut-outs of sectors that show 20^o or 30^o were prepared. The use of these cut-outs was, in the opinion of the teachers, helpful in assisting students construct the target mental representations.

The teachers started with the problem of showing different angles using the cut-out pieces. As they solved this problem, they found that the problem was too open. They had to decide on the number of each type of cut-outs to use. They had to decide on how the cut-outs were to be arranged – adjacent to each other, on top of each other or a combination of the two methods. For example, a 20^o piece and a 30^o piece can be placed side by side to show an angle of 50^o. A 20^o piece can also be placed on top of a 30^o piece to show an angle of 10^o. As they solved the problem, they also predicted that students would find the latter challenging, although the use of cut-outs may be useful.

According to the lesson plan, students were to be given a certain number of cut-outs of sectors that show 20^o or 30^o. In this lesson, students were asked to use (a) a 20^o piece and a 30^o piece, and (b) two 20^o pieces and a 30^o piece. Students were required to use these to show other angles.

The research lesson and a post-lesson discussion were conducted in the third session. One of the teachers taught the lesson to a primary four class. Each teacher observed one group of students. The teachers were reminded that they needed to observe students carefully to collect information on student thinking. The next part of this paper focuses on how this research lesson and the information collected helped the teachers identify factors that encouraged or discouraged mathematical thinking during a lesson.

The fourth session focused on revising the lesson plan based on the findings of the research lesson. Subsequently, another teacher taught the lesson to a different primary four class. A post-lesson discussion was again conducted. The final session was spent identifying parts of the lesson plan where mathematical thinking is prominent and delineating teacher actions that are able to stimulate, scaffold, encourage and perpetuate mathematical thinking.

FACTORS THAT ENCOURAGED OR DISCOURAGED MATHEMATICAL THINKING

The first research lesson was made of five main segments. Figure 2 describes the structure of the first research lesson which was 30 minutes long.

Segment	Description	Time
1	The teacher reviewed the idea of angles generally.	00 : 00
2	The teacher helped students understand the task for the lesson using one red (20^o) cut-out and one green (30^o) cut-out.	03 : 10
3	The students worked in groups using two green (30^o) cut-outs and one red (20^o) cut-out.	06 : 55
4	The teacher used two groups' solutions to lead a whole-class discussion.	12 : 00
5	The teacher conducted a general conclusion to the lesson	25 : 50

Figure 2: The structure of the first research lesson

The post-lesson discussion focused on the research theme – to develop a possible approach to help students construct a visual representation of angles with a particular emphasis on representations that are considered challenging to the students. The bulk of the post-lesson discussion was on the factors that encouraged or discouraged mathematical thinking. The following paragraphs are a synthesis of the post-lesson discussion.

The use of the cut-outs was critical in helping some students construct a visual representation of the central ideas of the lesson. This was particularly true in the challenging cases.

Many students did not face difficulty when the cut-outs were placed adjacent to each other. Thus, many students were able to see how a green piece and a red piece could show 50^o readily. The role of the cut-outs differed among different students in this situation. There were students to whom the cut-outs did not matter. They could say how 50^o could be shown without using the cut-outs. These students already had the visual representation of the idea and were making use of it to complete the tasks confidently. Then, there were students who used the cut-outs to strengthen their visual representation. They could say how 50^o could be shown but used the cut-outs to confirm their thinking. Finally, there were students who needed to use the cut-outs to arrive at the conclusion of how 50^o could be shown.

Many students had difficulties when the cut-outs were placed on each other. Thus, not many students were able to see how a green piece and a red piece could show 10^o by placing the red piece on top of the green piece in a certain way. The few students who could still needed the cut-outs to confirm their thinking.

The majority of the students needed the scaffolding provided by the teacher to make the cut-out useful in developing a visual representation of the idea. As the scaffolding was important, the teachers agreed that they needed to be more rigorous in developing the scaffolding questions. This was done for the second research lesson and the positive effects of carefully-constructed scaffolding questions were apparent.

The arrangement for students to work together in groups provided opportunities for students to encounter responses that differed from one's own. This led to students questioning their peers, seeking clarifications, defending their responses and resolving conflicting views. Such extended engagement with ideas was found to be conducive for mathematical thinking.

The use of the worksheet did not allow for such extended engagement. Answers had to be obtained and recorded promptly. In completing such a worksheet, the students were more eager to have an answer they can record to the teachers' satisfaction. There was little opportunity for engagement with ideas. It was decided that it would be better not to require students to complete a worksheet where answers had to be obtained and recorded quickly. In the second research lesson, the worksheet was not used. Instead, students were given an individual worksheet at the end of the lesson to consolidate the ideas that they had discussed in the lesson.

While the majority of the lesson was focused on a set of related problems, the first and last segments of the lesson were too general to be useful. General, superficial discussion of ideas did not facilitate mathematical thinking. On the other hand, students working on one problem that a set of solution ranging from obvious ones to challenging ones facilitated mathematical thinking. In the second research lesson, these segments were removed without affecting the main aims of the lesson. The time was used instead to complete the individual worksheet at the end of the lesson.

The problem used in the lesson was open enough to engage students in mathematical thinking. However, the teacher provided the suggestion that the pieces could be placed on each other even before the students had a chance to consider it. This premature direction robbed the students with a chance to make sense of the situation. In the first research lesson, there were students who simply placed the red piece on the green pieces without understanding its significance. This was because the teacher had said that the pieces could overlap. This suggestion was not given in the second research lesson. While fewer groups came up with this method of showing angles independently, these groups need no further help from the teacher in understanding its significance.

The information the teachers collected during the research lesson had resulted in teacher understanding of factors that facilitated mathematical thinking and those that were obstacles to mathematical thinking. Generally, the following was found to be a bridge to mathematical thinking: (a) the use of concrete material to anchor students' thinking, (b) the use of carefully crafted scaffolding questions to help student clear challenging situations a step at a time, and (c) extended engagement with ideas where students encountered different and, sometimes, conflicting views and where they had to question, clarify, justify and defend ideas. The following were found to be obstacles to mathematical thinking: (a) the use of worksheet that required a response to be recorded promptly, (b) the use of closed problems or the conversion of open problems to closed ones by providing directions too early in the problem-solving process.

LESSON STUDY IN DEVELOPING PEDAGOGICAL KNOWLEDGE

The data collected from this study involving eight teachers going through one lesson study cycle in helping teachers develop approaches to cultivate mathematical thinking amongst students allows a brief discussion on the use of lesson study in developing pedagogical knowledge.

In the lesson planning phase, solving the problems themselves allowed teachers to experience mathematical thinking and clarify to themselves what mathematical thinking means. In solving the problem in this study (finding angles that can be shown using a number of cut-outs that show 20^o and 30^o), one teacher very quickly realized the idea that "all multiples of twenty and thirty can be shown", to which another teacher extended when she said "so can all multiples of fifty". The former later included generalizing as an important part of mathematical thinking. Another teacher saw that 20^o can be shown by placing one cut-out on the other. He was made to clarify what he meant and to justify his thinking as several of his colleagues did not understand him. He included defending one's idea as an important part of mathematical thinking. In lesson study, the lesson planning stage included opportunities to reflect and articulate one's thinking in solving the problems selected for the lesson. In individual lesson planning, the reflection and articulation opportunities are left to chance.

In the research lesson phase, observing the students' thinking closely allowed teachers to see mathematical thinking in action. They are also able to see aspects of mathematical thinking that are easy for the students and those which are challenging. Teachers are also able to see instructional strategies that facilitate or inhibit mathematical thinking. In cases where the teachers have the opportunity to revise the lesson plan and conduct a second research lesson, they are able to test their conjectures. The research lesson also shows up instructional strategies that require more careful planning. In this study, the teachers initially did not realize the need to plan the scaffolding questions closely. As a result, the challenging part of the problem (the case of overlap) was not grasped by many students. In the revised lesson plan, the scaffolding questions were carefully crafted. This revised action bore positive effects in the second research lesson. The research lessons, thus, have the twin roles of showing the facilitating or inhibiting effects of instructional strategies including when these strategies are absent or not rigorously designed.

A RESEARCH AGENDA

In Singapore, professional development courses offered by the National Institute of Education are typically in the form of 24-hour courses. A new in-service course in the form of lesson study will be proposed. The structure of the course will be similar to the one described here with an initial session to introduce the lesson study process and a final session to allow teams to share their experience.

The research questions are (1) How do teachers develop their pedagogy in cultivating mathematical thinking amongst primary school students through lesson study? (2) What are the effects of lesson study on the teachers' mathematical thinking, perception of what mathematical thinking is and pedagogical knowledge of cultivating mathematical thinking?

Instruments will be developed to collect data for teachers' mathematical thinking and their perception of what mathematical thinking is. Changes in teachers' pedagogical knowledge will be based on field notes collected during the sessions to study the instructional materials, to plan and revise lesson, to discuss the research lessons and to identify specific points during a lesson where there is significant mathematical thinking and instructional strategies that support it.

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THE VAN HIELE LEVELS OF GEOMETRICAL THOUGH IN AN IN-SERVICE TRAINING SETTING IN SOUTH AFRICA

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ABSTRACT

This short presentation reports on an in-service training programme of Primary School teachers in a mining town in Mpumalanga, one of the nine provinces in South Africa. At this particular session, teachers were placed in a simulated classroom situation where they were exposed to the van Hiele levels of Geometrical thought. This session mainly concentrated on van Hiele level zero (visualisation). Various two dimensional shapes were provided to teachers in groups. The following procedures were followed:

- One teacher would choose a shape, and the rest of the group would then describe the shape
- The groups were asked to classify the shapes according to given properties

The videotape which will be shown, reveals most interesting thinking processes of teachers, which can be used fruitfully in any teaching environment.

In another video clip, an example of intervention that was not conducive for mathematical thinking will be shown, which can be used as a sample for discussion.

USING LESSON STUDY TO CONNECT PROCEDURAL KNOWLEDGE WITH MATHEMATICAL THINKING

Patsy Wang-Iverson, Gabriella and Paul Rosenbaum Foundation
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Developing U.S. students' mathematical thinking frequently is an elusive goal. The reasons are varied. Some of them include: 1. teachers' own lack of understanding of mathematics caused in part by an absence of a coherent mathematics curriculum (Schmidt et al., 2002) ; 2. insufficient or no professional development focused on the scope and sequence of mathematics within and across the grades; 3. inadequate knowledge and concrete examples of what mathematical thinking entails for both students and teachers; 4. lack of clear and explicit examples for how to connect students' procedural knowledge with conceptual understanding through mathematical thinking.

To focus APEC (Asia-Pacific Economic Cooperation) member economy specialists' attention on the importance of and approaches to the development of mathematical thinking of both students and teachers, the 2-7 December 2006 APEC lesson study conference in Tokyo/Sapporo, Japan, offered various keynote presentations (Katagiri, March, 2007, Lin, March, 2007, Stacey, March, 2007, Tall, March 2007). The speakers shared their perspectives on approaches to developing mathematical thinking, thus setting the stage for observation and discussion of four lessons, discussion of specialists' papers on mathematical thinking, and preparation for work following the conference. Prior to the end of the conference, the APEC member economy specialists were charged with the task of returning to their country and conducting a lesson study cycle that helped teachers work with their students to develop mathematical thinking skills while working on a specific mathematical concept.

Getting Started

To carry out the assigned task, the U.S. representative to the APEC lesson study conference (Wang-Iverson) invited the mathematics supervisor (Palumbo) at Bernards Township Public Schools (New Jersey) to identify a group of teachers willing to participate in lesson study. Although lesson study has been implemented at various sites across the United States since 1999[1], the team of five grade seven teachers (William Annin Middle School) that agreed to participate in the project was new both to lesson study and to discussing collaboratively how to develop students' mathematical thinking. Palumbo had engaged in lesson study previously with a few high school mathematics teachers, but she had not established a systemic lesson study initiative. The project thus

[1] Paterson School No. 2 was the first U.S. school to begin lesson study, under the tutelage of teachers from Greenwich Japanese School, a relationship facilitated by Makoto Yoshida (2004).

was two-pronged: introducing these individuals to the purpose, practice, and outcomes of lesson study and facilitating their collaborative work to develop student mathematical thinking through creating and teaching a lesson.

A unique feature of this district is that subject matter teachers at the same grade meet regularly to identify and discuss topics to be covered the following week and to share responsibilities for developing worksheets and homework assignments to be used across the classes. Although regular meetings are common in some districts, rarely do teachers progress at the same pace and share worksheets. What did not occur, prior to their engagement with lesson study, was observing each other's classes and discussing what was observed and what changes needed to be made to foster better student learning. These teachers' lack of opportunity to observe their peers and to be observed in turn is not an uncommon phenomenon across most countries, but U.S. teachers have even fewer opportunities. Internationally, 27% of grade 8 students participating in TIMSS 2003 had teachers who reported they had opportunities to observe colleagues two to three times a month; in the U.S. the number was 11%. Eighty-five percent of U.S. students had teachers reporting they have never observed or been observed by colleagues (Mullis et al. 2004).

During the introduction to lesson study, the teachers identified the characteristics of ideal students vs. the real students they encountered in their classes (see Appendix 1). This list was to serve as the basis for developing a lesson focused on moving students toward more idealistic behavior in learning mathematics. A common trait among students was their focus on simply getting the right answer and moving on to the next task.

Given the common schedule shared by the teachers, they next reviewed the topic they would be covering around the spring dates selected for teaching the lesson. However, after observing some classes on proportions and noting students' tenuous grasp of the concept, the authors suggested the teachers might wish to revisit the concepts of percent and proportion and create a lesson that helped strengthen students' understanding of those topics.

Facilitated by the mathematics supervisor, the teachers developed a lesson study schedule that allowed them to meet weekly or biweekly to plan and develop the lesson. The supervisor suggested the lesson study group use moodle (www.moodle.com), a web-based platform, to document conversations between meetings, which then could be archived. However, this practice was not maintained throughout the lesson study process, as it introduced yet another new undertaking for the teachers in the midst of continuing their regular work of daily teaching.

After a very brief introduction to kyozaikenkyu[2] (Takahashi et al., 2005), the teachers investigated various resources to find problems they wanted to use in their lesson. Through their individual exploration of resources beyond just the textbook, the teachers selected problems that might push student thinking about fractions, percents and proportions. They then reviewed the problems and ranked them. At a subsequent meeting they discussed the merits of the problems selected and agreed upon one problem, which is the first problem presented in the lesson plan (Appendix 2):

Problem: The Carters are buying a new iPod Nano. Three stores have on sale this week the model they want, but they have decided to shop at Ralph's, because they think Ralph's is offering a "double discount." Here are the ads. Did the Carters make a wise decision? Explain.

Radio Shop Original Price $172 Discount: ¼ off	Discount City Original Price: $180 Discount: 30% off	Ralph's Original Price $180 Discount: 10% off with an additional 20% off the discounted price

Identifying the goals was not a simple task; such an approach previously had not been the norm in preparing a lesson. The goals elucidated in the final lesson plan for the Algebra I lesson were focused more on the specific skills rather than on developing students' mathematical thinking:

- **a.** Understand the value for using efficient methods when solving percent problems
- **b.** Compare and contrast the relationship(s) between determining the "part" and "determining the "whole" in a percent problem

Developing and teaching the lesson

The lesson plan evolved over several meetings. One teacher volunteered to work on writing the rationale for choosing the particular lesson problem, another focused on writing the lesson plan itself, and the supervisor and one of the teachers who also taught grade six developed the scope and sequence of concepts taught in the earlier grades (see lesson plan in Appendix 2). However, in the effort to move on to develop the lesson plan,

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[2] "investigation of instructional materials," encompassing not just textbooks, teacher manuals, and mathematics manipulatives, but a wider range of materials, including the course of study (standards), the educational context, learning goals, tools, research and case study publications, lesson plans and reports from lesson study open houses, and ideas gained from research lesson observations. *Kyozaikenkyu* also includes investigation of students' prior knowledge, learning experiences, state of learning and understanding, which makes it possible for teachers to be able to anticipate students' reactions and solutions to the problems students study during the lesson.

the scope and sequence and the accompanying text pages were not studied in detail and discussed by the group in planning the lesson.

Initially, the teachers seemed reluctant to volunteer to teach the lesson, but at the next meeting, they all expressed a desire to teach, as they were interested in having their colleagues observe their students. It was agreed that the lessons would go through two paired iterations in grade 7 classrooms followed by a final iteration in a grade 7 algebra classroom (see table). Two teachers would conduct the first teaching (1-1 and 2-1); to avoid being influenced by the first lesson, the second teacher would not observe the first lesson. This format did not follow the usual lesson study process, where one teacher volunteers to teach the lesson, followed by discussion and revision of the lesson. Whether teaching of the revised lesson takes place varies across lesson study groups.

The structure adopted for this lesson study project provided more opportunities for the teachers to practice their observational skills focused on student thinking and learning. Following the lessons, the team met to share and discuss the data collected and to revise the lesson for the second teaching by two more teachers.

During the first teaching the students simply had been asked to solve the problems on the worksheet. For the second teaching, students received a worksheet that provided room for them to solve the problem in more than one way and to record the time when they finished the problem (see Fig. 1). The time recorded by the students provided useful data for the teachers for scheduling more effectively in the future the amount of time needed by students to complete assigned tasks. The students in these two classes were asked to write down their reflections on a form containing specific questions (see Appendix 2). After the second teaching, the lesson was revised again and re-taught by the teachers in their other classes without observers.

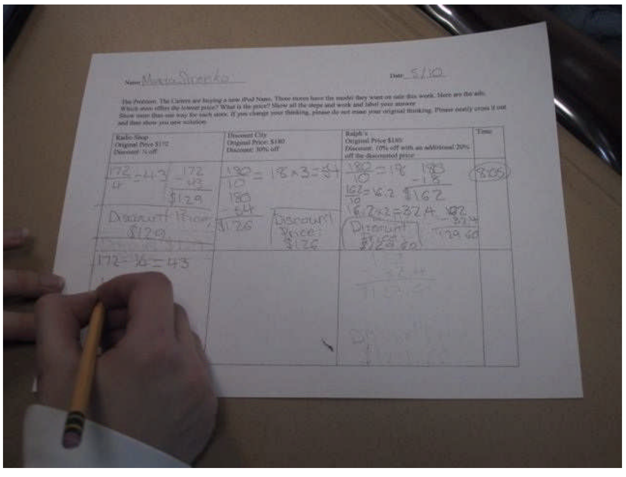

Figure 1. Student worksheet

Table: Teaching sequence for observed lessons

	First teaching	Second teaching	Third teaching
Teacher #1	1-1		
Teacher #2	2-1		
Teacher #3		3-2 (with calculator)	
Teacher #4		4-2 (w/o calculator)	
Teacher #5			5-3 (algebra)

Investigating additional factors

Two teachers used the lesson to investigate how other factors affect student thinking:

1) Calculator usage: One of the teachers during the second teaching of the lesson did not give students calculators to use during the lesson, which provided an opportunity for observers to analyze differences in student work and thinking with/without the use of calculators.

2) Advanced students: One of the teachers who taught a grade 7 algebra class in addition to regular grade 7 mathematics classes further modified the lesson and taught it to the algebra class (final teaching of the lesson). She was able to assess the differences in mathematical thinking between the grade 7 regular mathematics students and her grade 7 algebra students, who were considered more advanced mathematically.

Documentation of student work

During the first teaching of the lesson, some students attempted to solve the problem using proportions but set up the problem incorrectly (see Fig. 2):

25/172 = x/100; 30/180 = x/100; 10/180 = x/100

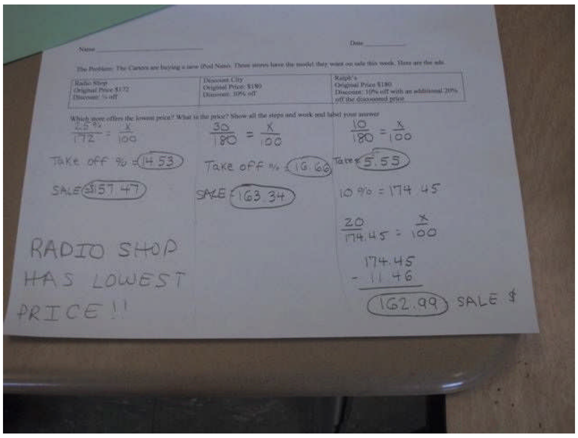

Figure 2: Error using proportions

As a result, students were not in agreement on which discount provided the lowest price. For students who arrived at the correct conclusion that Discount City provided the largest discount, they first calculated the discount and then subtracted the value from the original price:

Given: $180 = original price; 30% discount

180 x 0.3 = 54
180 – 54 = $126 = discounted price at Discount City

No students in these two classes solved the problem by directly calculating the fraction or percent of the original price:

180 x 0.7 = $126 = discounted price at Discount City

In the second iteration of the lesson, when students were asked to solve the problem in more than one way, seven out of 25 students in one class subtracted ¼ from one and found ¾ of $172 to calculate the discount price at Radio Shop. In the other class, only one student used this method to solve the problem. Most of the other students, in trying to find a different method, moved between multiplying by a fraction and multiplying by the fraction's decimal representation, considering these to be different solution methods. Six of the 44 students in the two classes did not show a second way of solving the problem. These results implied the students may not have been used to being asked to solve problems by more than one method, and some did not understand what it meant to think about tackling the problems in different ways.

Confronting a more challenging problem

During the first teaching of the lesson (1-1 and 2-1), no student was able to solve the additional problem, which asked for the original price of the computer, given the discounted price:

Additional problem: A computer is discounted 20% from its original price because it didn't sell. The store took an additional 30% off the discounted price. Barbara purchased the computer for $896. What was the original price of the computer?

Three students in one of the classes during the second iteration of the lesson obtained the correct original price: one student had the correct calculator-generated answer but no written record, while the other two students solved the problem using the following steps:

100% - 30% = 70%
896 ÷ 70/100 = 896 x 100/70 = 89600/70 = 1280
100% - 20% = 80%
1280 ÷ 100/80 = $1600

In using the above steps, these students were able to apply the knowledge used in the earlier problem (subtracting the discount from 100%), but they needed to go one step further to realize that in order to calculate the original price, they needed to divide rather than multiply. Students who were not able to solve the problem correctly did make valiant efforts, trying to apply what they had learned previously. Some set up the proportion formula, $a/b = p/100$ (taught earlier in the year by the teacher from the textbook), but then did not know what to do next, demonstrating they remembered but did not understand the formula (Stacey, 2007, p. 45). Students fell into the trap of either multiplying the discounted price by the percent discount ($896 x 0.3) or dividing by the percent discount ($896 ÷ 0.3). Other students multiplied by 0.8 and 0.7. A few students knew to divide by 70% but then divided by 7 and not 0.7. Further analysis and conversation with the students might have helped to determine whether this error is merely computational in nature or reveals a more fundamental problem in moving from percent to decimal notation.

One student arrived at an answer of $949.76 by the following route:

896 x 0.3 = 268.8
268.8 x 0.2 = 53.76
896 + 53.76 = $949.76

The student incorrectly applied the strategy used in the earlier problem (Ralph's store): sequential multiplication. In this case, seeing that $53.76 could not be correct, since the original price had to be greater than the discounted price, s/he then simply added this value to the final discounted price to arrive at the 'original' price. This solution illustrates the student's tenuous grasp of the earlier solution method, leading to an inability to apply it to a different problem.

Another student obtained an answer of $1396.96 using the following method:

896 x 0.3 = 268 4/5
896 + 268 4/5
1164.8 x 2/10 = 252.96
1164 + 232.96 = $1396.96 = original price

In addition to moving between the use of fraction and decimal in solving the problem incorrectly, this student also tried to apply directly what was previously discussed for a different problem to find the discounted price. Perhaps in an effort to compensate for the difference between the two problems, in lieu of subtracting, the student added to arrive at the original price. In this case it would have been useful to ask the student to explain the thinking behind the calculations.

The above two examples illustrate students' readiness to 'push buttons' to arrive at an answer but an inability to evaluate the work to make sense of the calculations. In developing mathematical thinking, students need to learn to slow down and to be taught explicitly how to engage in metacognition, scrutinizing one's own thinking. This lesson study cycle revealed the need to help students move beyond simply applying algorithms without considering whether they make sense for solving the specific problems.

In the advanced class there were no computational errors. Twelve of the 16 students solved the iPod problem by first calculating the discount and then subtracting it from the original price; the remaining four students directly calculated the discounted price for the iPod problem. These same four students calculated the price at Ralph's using a two-step process: first calculating the 10% discount followed by the 20% discount.

One student in this advanced class initially calculated the answer for Ralph's by the following method:

180 x 0.1 = 18
180 – 18 = 162
162 x 0.2 = 32.4
162 – 32.4 = $129.60

From this solution, she then was able to reduce the steps to one equation:

x = (180)(0.9)(0.8) (see Fig. 3).

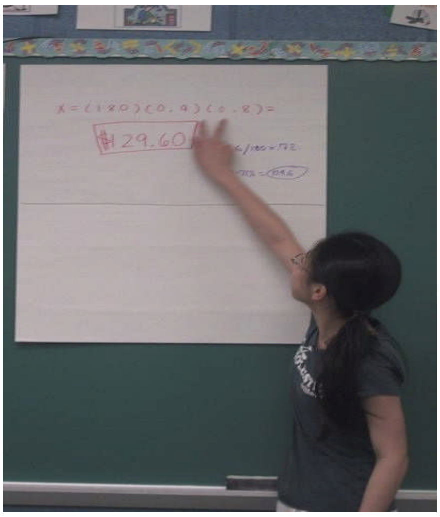

Figure 3: Expressing the iPod problem in one step.

After the student presentations, the teacher summarized the approach to using a one-step equation to finding the cost of the iPod at Ralph's. She then asked the students to solve the computer problem. The majority of students did not solve the problem correctly; in that time period they were not able to transfer what they had learned from the previous problem. A few students were able to solve the problem by first dividing by 0.7 and then dividing by 0.8, which was a solution anticipated by the teacher (see lesson plan in Appendix 2). One student reduced the equation to:

Original price = 896 ÷ 0.56 = $1600

For most of the students, using the more efficient method of solving for a double discount presented a new way of thinking about the problem. In hindsight, perhaps they needed to have first solved a problem asking for the original price after a single discount and then move on to the computer problem with its two discounts[3].

Student discourse

Students worked in pairs in all the classes. Three patterns of behavior[4] were observed: 1. although asked to work with a partner, students worked silently and individually; 2. one student immediately took charge and told the other student what to do; 3. the two students worked as a team, discussing their answers as they worked.

Two students in particular during the third teaching (5-3) carried out a prolonged discussion of their answer of $14,933.30 for the computer problem, which they had obtained by dividing the sale price of $896 by 0.3 and then by 0.2. The original price they obtained seemed too high to them, but in checking it, using the same decimals, they came up with the same number. They concluded the answer had to be right, despite feeling perplexed by the large number. Neither student questioned the validity of their thinking; they simply checked their calculation without considering that perhaps they were using the wrong numbers.

Another pair of students in class 4-2 engaged in a debate over what one student had written for the proportion they had set up. The second student insisted the first student's work was wrong, while the first student replied that what she had written was correct. The first student finally understood the source of the second student's disagreement and said that the proportion she had set up was correct, but that she simply had written it as $p/100 = a/b$, rather than $a/b = p/100$, which was the standard way shown by the teacher and the textbook. This exchange revealed that one student understood the formula (understood that the two sides of an equal sign can be exchanged without changing the

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[3] Japanese lesson 3 from the TIMSS Video Study (www.rbs.org/international/timss/resource_guide/lessons/by_country.php#japan) was an introduction to inequalities. After the teacher summarized the student solutions, he then presented a second, easier problem, which would allow all students to solve it using inequalities.

[4] From this behavior, it appeared some students did not understand the benefits of working with a partner (Gould, 2007), and there might not have been whole class discussion of the purpose of working collaboratively.

relationship), while the other student simply remembered the formula (Stacey, 2007, p. 45).

According to Gould (March, 2007), “Learning to argue about mathematical ideas is fundamental to understanding mathematics.” To be prepared to argue, students need to be able to listen to and respond to each other’s explanation of their work and thinking. The above issue was resolved, because the first student was able to listen to and understand her partner’s point of dissension.

Mathematical thinking

Although the term ‘mathematical thinking’ is used over 100 times in the *Principles and Standards of School Mathematics* (NCTM, 2000), no clear and explicit definition is provided. Stacey (March, 2007, pp. 39-40) described mathematical thinking as a “highly complex activity”, a process “…best discussed through examples.” Katagiri (March, 2007) also does not provide a clear definition, but he illustrates the logical steps (in order of complexity) of mathematical thinking for a counting problem that he used as an example (p. 115):

- Clarification of the meaning of the problem
- Coming up with a convenient counting method
- Sorting and counting
- Coming up with a method for simply and clearly expressing how the objects are sorted
- Encoding
- Replacing with easy-to-count things in a relationship of functional equivalence
- Expressing the counting methods as a formula
- Reading the formula
- Generalizing

Mathematical thinking is “the most important ability that arithmetic and mathematics courses need to cultivate in order to instill in students this ability to think and make judgments independently (p. 108)…“To be able to independently solve problems and expand upon problems and solving methods, the ability to use “mathematical thinking” is even more important than knowledge and skill, because it enables to drive the necessary knowledge and skill (p. 110). A working group composed of computer scientists and mathematicians offers a very general definition of mathematical thinking as “applying mathematical techniques, concepts and processes, either explicitly or implicitly, in the solution of problems.” (Henderson et al., 2001).

The ability to think and make judgments independently has been the goal of Japanese education since 1950, but it still remains to be achieved (Katagiri, p. 108). Such is the case also in the United States. As U.S. teachers turn to lesson study in mathematics to help them develop the ability to better understand and analyze student thinking and learning, they are finding they first need to understand how the students are thinking (or not thinking) about the mathematics they are being taught and then learn to move students from simply following and applying procedures in very rigid and limited ways to developing the ability to determine for themselves which procedures to use, how to achieve a level of efficiency in solving the problems, and whether what they have done makes sense.

Key window for considering mathematical thinking

The key window in this lesson study was communication, at the levels of teacher-to-teacher, teacher-to-student and student-to-student communications. In planning for the first teaching, there was no discussion of solution efficiency, and anticipation of student thinking and misunderstanding was limited. After observing the first teachings, the teachers discussed the need to probe more deeply students' understanding of the problem by offering counter-examples[5] to student solutions to push their thinking. For the second teaching it was agreed that students would be asked to consider if a 10% discount followed by a 20% discount was the same as or different from a 20% discount followed by a 10% discount. The students would also be urged to support their answer mathematically. After the two iterations of teaching, the teachers also began to focus on the need to help students consider how to solve problems by looking for student-generated efficient solutions and discussing them as a whole class.

At the level of student-to-student communication, the teachers began orchestrating more carefully the sharing of the student solutions, encouraging the students to communicate their solution strategies in a sequential fashion in order to enhance student understanding. The student presentations were planned to flow from the concrete to the abstract, from specific to general, from "ordinary solutions" to "efficient solutions." This teaching strategy was learned from watching a TIMSS video of a Japanese teacher orchestrating the student solution process (Hiebert, et al., 2003) prior to beginning the lesson study cycle.

What did teachers learn?

Subsequent to this first experience with lesson study, the teachers now report that in planning lessons, they think more carefully about anticipating students' solutions and orchestrating the manner in which the students communicate the solutions to the other members of the class. This is a change from the process previously in place, in which the teachers randomly selected students to come to the board to explain a solution to the problem. When the teachers used this practice (random selection versus planned selection

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[5] For calculating Ralph's discount, the teacher might ask why one couldn't first add 10% and 20% and then multiply the original price by 30%.

of student solutions), the flow of the lesson could be interrupted by "surprises" that could also confuse or misdirect students away from the learning objective.

Teachers reflected upon this first lesson study experience by responding to a series of questions (see Appendix 4). One of the main impacts of the lesson study cycle was to strengthen the teachers' ability to examine students working in the classroom and to discuss their observations, in turn making the teachers themselves more reflective thinkers, as documented in their questionnaire responses. Through the eyes of their colleagues, they learned more about their students' thinking; they obtained information about students beyond what was written on the student worksheets. Questions posed by colleagues during the post-lesson discussion caused them to rethink the approaches, activities and worksheets they used. Most importantly, the questions allowed them to consider the lesson and whether all that was planned and done really contributed to achieving the goals of the lesson.

Through practice made possible by all the teachers volunteering to teach the lesson, they became more proficient at observing lessons and collecting data on student thinking. Additionally, two teachers commented that due to their experience with lesson study they more carefully choose problems for both discussion and practice, look closely at the wording in selected problems to eliminate any ambiguity, and will better plan the sequence of problems on any future worksheets.

During the planning phase of the lesson study cycle, there was no detailed discussion of the scope and sequence (what students had learned in previous grades), accompanied by examination of the elementary textbooks and curriculum guide. However, it did highlight the teachers' previous strict adherence to the textbook, which in turn precipitated a subsequent review of the scope and sequence of the district's mathematics curriculum and the recognition of the need to align it with NCTM's *Focal Points* (NCTM, 2006). They recognized the need to use the "book more as a tool to help achieve the goal of the lesson and not to let the book become the goal." They also realized it was necessary to consider what students might have learned in previous years, how the concepts were taught, and what language was used in order to build upon students' prior knowledge and to understand the root of students' confusion.

Another realization was the need to move away from telling students too much to giving students an opportunity to come up with their own solution methods. To quote one teacher, "For true learning by the students, they need to be able to make or to see connections between what they already know and what it is we are trying to teach them." One teacher identified the lesson study process as an assessment tool that helps teachers see what students know about a topic and what knowledge they lack (misunderstanding).

Conclusions

Observers in classrooms often hear teachers ask students to "think." Sometimes it is not clear about what and how students should be thinking. The APEC lesson study project, recognizing the intricacies in developing mathematical thinking, has devoted a series of conferences to the discussion of this very important topic. Observations of Japanese classrooms reveal the deliberate and explicit ways by which teachers help students learn and develop mathematical thinking skills; no steps are skipped, and no assumptions are made about student understanding.

Developing students' mathematical thinking requires a coordinated group effort, as exemplified by the lesson study process. Teachers learn from colleagues' data collected from observation of their students. The purpose of lesson study, however, is to inform daily instruction, when teachers are alone in the class with their students. By providing teachers with the opportunity to teach in front of colleagues and to collect data on student learning, thinking, and misunderstanding in colleagues' classrooms, lesson study focuses teachers' attention on how students interpret or misinterpret the lesson. Better understanding of students' thinking can help teachers develop lessons that build students' understanding rather than cause or contribute to their confusion.

Many teachers' goal is to develop lessons that flow smoothly. However, a lesson that unfolds exactly as orchestrated may not shed light on real student thinking and understanding. The students in this lesson study cycle revealed to us a great deal about their misunderstandings and tenuous grasp of concepts, providing us with crucial information on the necessary next steps to correct their misunderstanding and to provide the scaffolding needed to build their understanding.

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Appendix 1: Ideal Students vs. Real Students

Ideal Students

* are prompt, polite, and prepared
* are respectful of each other and teacher
* persevere
* are motivated, interested, engaged
* are self-starters
* are self-reflective; engage in meta-cognition
* are active members of classroom discussions
* are responsible for their own learning
* take pride in their work
* are honest, have integrity

Real Students

* are unprepared: no tools, homework, mentally
* lack perseverance
* are unable/unwilling to think through problems
* are impulsive; act without thinking
* have a “tell me how to do it” attitude; just want to get it done
* don’t take time to assess reasonableness of answer
* exhibit varying levels of interest and perseverance within each classroom
* display lack of understanding
* have no concern for quality work
* are bored?
* are too motivated by grades
* don’t show thinking in writing (due to laziness?)

Were these lists written by the teachers in an effort to vent? Do they have real steps for turning their real students into ideal students?

Appendix 2:

Lesson Study at William Annin Middle School

Lesson study team members:
Patricia Gambino , Tara Gialanella, Chad Griffiths,
Mary Henry, Marian Palumbo, Elizabeth Slack

1. **Title of lesson: Assessing Student Understanding of Percent Concepts**

2. **Lesson Goals:** Students in grade 7 Algebra I from William Annin Middle School will:

   a. Understand the value for using efficient methods when solving percent problems
   b. Compare and contrast the relationship(s) between determining the “part” and “determining the “whole” in a percent problem

Class organization: Students will work with a partner to solve the problem. One student from selected pairs will put the solution on the board.

Rationale

Initially, as we worked with our seventh-grade students, we all became aware that our students did not have a deep understanding of the concept of percent. Moreover, it was clear that many students did not see the connection between fractions, decimals, percents, and proportions. Therefore, we decided to reexamine this concept. We felt that we needed to assess our students’ current grasp of the topic of percent and uncover the sources of their misunderstandings and why they are not making the connections. It was at this point that it became clear that this topic, not the one we had originally chosen, should be the focus of our lesson study. Therefore, we decided that we would present our students with three problems involving percents and sale prices. Our students would have to decide at which store to buy an iPod in order to pay the lowest price. We chose this scenario, because we thought that it would grab our students’ interests and be familiar to them. In addition, knowledge about and facility with percents is an important life-long skill.

Initially we presented the lesson to our regular seventh-grade mathematics students, some of whom did not find the lesson particularly challenging, as they were applying the same rote procedures they had learned in earlier grades. When we revised the lesson for the seventh-grade students enrolled in Algebra I, we realized we needed to give them additional opportunities to compare and contrast the various types of percent problems and to focus their attention on using efficient methods for solving the problems.

Scope and Sequence for Fractions, Decimals, Percents

Grade 2	Grade 3	Grade 4	Grade 5	Grade 6
Fraction equivalences (informal exploration)	• Fraction equivalences continued • Decimal concept introduced	• Operations with fractions/decimals introduced • Comparing, ordering decimals • Adding/subtracting decimals • Fraction concepts • Adding/subtracting fractions • Percents introduced • Convert fractions to decimals and percents w/ calculator • Multiply/divide decimals	• Add, subtract fractions • Multiply fractions using area model • Relate fractions, decimals, percents • Convert fractions to decimals, percents • Find percent of a number • Use unit fractions to find the whole • Use percents to interpret/create circle graphs	• Convert between fraction, decimal, percent • Review finding percent of a number • Use proportions to solve percent problems • Application: calculate tip, discounts, and sales tax

The Problem: The Carters are buying a new iPod Nano. Three stores have on sale this week the model they want, but they have decided to shop at Ralph's because they think Ralph's is offering a "double discount." Here are the ads. Did the Carters make a wise decision? Explain.

Radio Shop Original Price $172 Discount: ¼ off	Discount City Original Price: $180 Discount: 30% off	Ralph's Original Price $180 Discount: 10% off with an additional 20% off the discounted price

The management at Ralph's decided to change their ad to attract more customers. Here is the new ad:
Original Price $180. Discount 20% off with an additional 10% off the discounted price. How does this change the sale price? Explain.

3. Lesson Plan

Time	**Teacher Activity**	**Anticipated Student Thinking and Activity**	**Point To Notice and Evaluate**
0-3 min	Set up the problem and check for student understanding • Teacher discusses each store one at a time, displaying props • Clarify any student misunderstanding or questions		
3-8 min.	Tell students they should work on the problem with their partner. Both students are responsible for showing the solution strategies on their individual sheets of paper. When both students are finished they can begin work on the additional problem (different colored sheet of paper). • Teacher circulates to identify various solution methods for Ralph's only and	Ralph's .1 (180) = 18, 180 – 18=162 162 (.2) = 32.4, 162-32.4 = 129.60 order unimportant? .9 * .8 * 180 = 129.60 .72 (180) = 129.60	What distribution of students used different methods?

Time	**Teacher Activity**	**Anticipated Student Thinking and Activity**	**Point To Notice and Evaluate**
	selects students to record work on board. • While circulating, distribute second set of problems to be completed when students finish initial problem, (distribution method is optional)		
8-11 min	Poll students by show of hands How many think the Carters made a wise decision choosing Ralph's? How many think the Carters made a poor decision? Why might the Carters think that Ralph's would have the lower sale price? • Facilitate a class discussion	The ad is misleading The words say a 10% discount followed by a 20% discount, which means that first you have to multiply by 10%, find the sale price and then find the 20% discount from the sale price. A 10% discount followed by a 20% discount is not the same as a 30% discount – that is what the Carters were thinking.	What comments do students make about the Carters decision? What distribution of students thought the Carters made a wise decision? What distribution of students thought the Carters made a poor decision?
11-20 min	Students present their solutions for Ralph's • Teacher calls attention and facilitates a short discussion about the more "efficient solutions" • If necessary introduce solution .9(.8)(180) and .72(180)	.1 (180) = 18, 180 – 18=162 162 (.2) = 32.4, 162-32.4 = 129.60 .9 * .8 * 180 = 129.60 .72 (180) = 129.60	What distribution of students used efficient methods? What distribution of students
20-22 min.	Introduce the new ad – (show) The management at Ralph's	It doesn't change Multiplication is commutative	What distribution of students demonstrates

Time	Teacher Activity	Anticipated Student Thinking and Activity	Point To Notice and Evaluate
	decided to change their ad to attract more customers. Here is the new ad: Original Price $180, Discount 20% off with an additional 10% off the discounted price. How does this change the sale price? Explain • Facilitate a short discussion	.9 (.8) (180) = .8(.9)(180)	application of the commutative property to this example?
22-30 min.	Call the students' attention to the additional problem and have them continue to work on that one (computer problem): A computer is discounted 20% from its original price because it didn't sell. The store took an additional 30% off the discounted price. Barbara purchased the computer for $896. What was the original price of the computer? • Teacher circulates to collect solutions	896/.7/.8 = 1600 896 /.56 = 1600	What distribution of students used an efficient method? What distribution of students was able to transfer their knowledge of the first problem to this problem?
30-36 min	Teacher facilitates a discussion about the various solution methods and then summarizes by comparing and contrasting both problems, and generalizing .9 (.8) (180) = x .9 (.8) (whole) = part .8(.7) (x) = 896 .8 (.7) (whole) = part		

Time	Teacher Activity	Anticipated Student Thinking and Activity	Point To Notice and Evaluate
36-40 min	Teacher closes the lesson, asking the students to reflect on their learning and then complete the questions on the reflection sheet,		

Name ________________________________ Date___________________

The Problem: The Carters are buying a new iPod Nano. Three stores have on sale this week the model they want, but they have decided to shop at Ralph's because they think Ralph's is offering a "double discount." Here are the ads. Did the Carters make a wise decision? Explain.

Radio Shop Original Price $172 Discount: ¼ off	Discount City Original Price: $180 Discount: 30% off	Ralph's Original Price $180 Discount: 10% off with an additional 20% off the discounted price

Name:______________________ Date:_____

Try This One!!!

1. A computer is discounted 20% from its original price because it didn't sell. The store took an additional 30% off the discounted price. Barbara purchased the computer for $896. What was the original price of the computer?

Name __________________ Date ______________

What mathematics did you learn or think about today?

In what ways was the lesson challenging?

In what ways was the lesson interesting?

Appendix 3: Template for data collection

Collection of Student Thinking

	Radio Shop Original Price $172 Discount: ¼ off	Discount City Original Price: $180 Discount: 30% off	Ralph's Original Price $180 Discount: 10% off with an additional 20% off the discounted price
Use multiplication with decimal to find the discount and then subtract			
Use multiplication with fraction to find the discount and then subtract			
Use multiplication with (100-x)%			
Use multiplication with (whole-part) as a fraction			
Use a proportion to solve the problem with x/100			
Use a proportion to solve the problem with (100-x)/100			
Non-solutions for Ralph's – addition of percents			
Other solutions (note solution)			

Appendix 4: Assessing teacher learning during lesson study

1. Reflect on 1-3 things you learned from the lesson study experience.

2. What did you do prior to lesson study that hampered student learning?

3. What changes might you make to enhance student learning?

4. In what ways have you deepened your own understanding of mathematics?

5. What did you learn from observing colleagues' classrooms?

6. What did you learn from your colleagues' observation of your students?

7. What has changed since the lesson study cycle?

A LESSON THAT MAY DEVELOP MATHEMATICAL THINKING OF PRIMARY STUDENTS IN VIETNAM

FIND TWO NUMBERS THAT THEIR SUM AND A RESTRICTED CONDITION ARE KNOWN

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In Vietnam, after launching the national standard mathematics curriculum in 2006, the classroom mathematics teachers have learnt more on the innovative teaching strategies to implement more effective lessons focusing on mathematical thinking. The aim of this paper is to examine a lesson that we considered may develop mathematical thinking of primary students in Vietnam. A case study will be analysed using the observed students' activities in a videotaped lesson.

INTRODUCTION

In Vietnam, teachers encourage their students to invent their own procedures or algorithms for solving problems. The teachers use the teaching strategies that aim to:

- Promote active, initiative and self-conscious learning of the learners;
- Form and develop the ability of self-study;
- Cultivate the characteristics of flexible, independent, and creative thinking;
- Develop and practice the logical thinking;
- Apply problem solving approaches;
- Apply mathematics to real life situations.

In the teachers' guidebook for primary mathematics teachers at each grade there are four main activities in a lesson that teachers should follow to develop mathematical thinking:

Activity 1. Teacher manages students to work and achieve the following aims:

- Examine the students previous knowledge;
- Consolidate the previous knowledge involved with new lesson;
- Introduction to the new lesson.

Activity 2. Teacher facilitates students explore mathematical knowledge and construct new knowledge by themselves.

Activity 3. Students practice the new knowledge by solving exercises and problems in the textbook.

Activity 4. Teacher concludes what students have learnt from new lesson and assigns the homework.

Engaging to the lesson, the pupils will have opportunities to show their mathematical thinking through:

- The ability of observing, predicting, rational reasoning and logical reasoning;
- Knowing how to express procedures, properties by language at specific levels of generalization (by words, word formulas);

- Knowing how to investigate facts, situations, relationships in the process of learning and practicing mathematics;
- Developing ability on analyzing, synthesis, generalization, specifying; and starting to think critically and creatively.

In our point of view, the key windows for considering mathematical thinking are as follows:

- Students learn mathematical concepts with *meaningful understanding*;
- Students construct *individual algorithm* and *techniques* themselves with understanding to solve some specific problems;
- Students use learnt mathematics to *solve* mathematical problems effectively;
- Students show mathematical thinking by *communicating* (talking, writing, arguing, discussing, and representing);
- Students *reflect critically* their mathematical thinking in order to improve their learning;
- Mathematical thinking is *social* and *relative* to each individual student;
- Students apply *logical* and *systematic* thinking in mathematical and other contexts;
- Students use *thinking operations* in solving problems: *comparison*, *analogy*, *generalization*, and *specialization*;

The lesson which will be analyzed in this paper is prepared by classroom teacher for grade five primary students. We can find from the lesson plan the three main tasks and a quiz proposed in the lesson:

Introductory Task. Use 2 cm-cards and 4 cm-cards to make a toy train of 5 wagons?

Task 1. Use 2 cm-cards and 4 cm-cards to make a train with the length of 16 cm?

Task 2. A train with the length of 50 cm including 20 wagons, how many red wagons and blue wagons are there?

Task 3. A train with the length of 100 cm including 36 wagons, how many red wagons and blue wagons are there?

Quiz (Homework). There are 33 liters of fish sauce contained in 2 liter-bottles and 5 liter-bottles. The number of bottles used is 12. Find the number of 2 liter-bottles and 5 liter-bottles used. Given that, all bottles are full of fish sauce.

At the end of Grade 4, students know how to solve and express solutions of problems having three operations of natural numbers.

Example. A toy train has 3 wagons with the length of 2 cm, and 2 wagons with the length of 4 cm. Find the length of the train?

Answer. $3 \times 2 + 2 \times 4 = 14$ (cm).

But in the second semester of grade 5, if we set the problem in a reverse way:

A toy train has two types of wagon: 2 cm-wagons and 4 cm-wagons. This train has the length of 14 cm including 5 wagons. Find the numbers of 2 cm-wagons and 4 cm-wagons of the train.

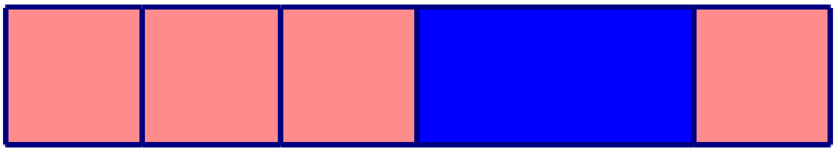
The sum of two numbers needed to find is 5.

Restricted condition: The total length of 2 cm-wagons and 4 cm-wagons is 14 cm.

This reverse problem is quite different with what students have learnt in grade 4. The problem requires them to analyze a natural number into sum of two other numbers satisfying a restricted condition logically.

ANALYSIS OF THE TASKS

Analysis of Introductory Task. Use 2 cm-cards and 4 cm-cards to make a toy train of 5 wagons?



This task is an introductory activity. It is an open-ended task that requires pupils to make many trains as possible. Pupils can arrange the cards to make a train, use the strategy "*guess and check*" to get many answers. To solve this task mathematically teacher guides students to make a systematic list of all abilities.

N. of red wagons	0	1	2	3	4	5
N. of red wagons	5	4	3	2	1	0
The length of the train in cm	20	18	16	14	12	10

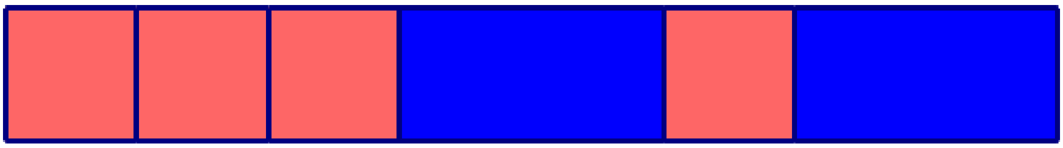
From the above table, students recognize the relationship between the length and the numbers of red wagons, blue wagons. If the number of red wagons increases one, then the length of the train decreases 2 cm. In this task, students know that:

N. of red wagons + N. of blue wagons = 5

There are 6 options for this task. If the length of the train is given then we can find exactly the N. of red wagons and N. of blue wagons. The length of the train is understood as a restricted condition. Students will see that the train has the longest length 20 cm when all of the wagons are blue and shortest length 10 cm when all of the wagons are red.

The aim of this introductory task is to help students recognize the restricted condition in finding two numbers that their sum is known.

Analysis of Task 1. Make a train with the length of 16 cm.



This is also an open-ended task that requires students to make a systematic list of all abilities. The restricted condition is given but the sum of two numbers is unknown.

N. of red wagons	8	6	4	2	0
N. of blue wagons	0	1	2	3	4

Total of wagons	8	7	6	5	4

There are 5 answers to this task. Students know how to analyze a natural number into sum of two natural numbers with a specific restricted condition.

$16 = 8 \times 2 + 0 \times 4$ $16 = 4 \times 2 + 2 \times 4$ $16 = 0 \times 2 + 4 \times 4$

$16 = 6 \times 2 + 1 \times 4$ $16 = 2 \times 2 + 3 \times 4$

From the table students will see that a train with the length of 16 cm including 6 wagons has 4 red wagons and 2 blue wagons. The restricted condition of this problem is:

Sum: N. of red wagons + N. of blue wagons = 6.

Restricted condition: The length of the train is 16 cm.

If all wagons are red then the length of the train decreases: $16 - 6 \times 2 = 4$, then the number of blue wagons: $(16 - 6 \times 2) \div 2 = 2$.

Students practice this procedure to consolidate what they have learnt. The most important fact that the students need to realize is the difference 2 cm between one blue wagon and one red wagon.

Analysis of Task 2. A train with the length of 50 cm including 20 wagons, how many red wagons and blue wagons are there?

In this task, teacher does not ask students to make a table but encourages them to generalise what they have observed in some concrete situations above to create their own procedure to solve the general problem.

Students make temporary assumption: If the train has only red wagons, the length of the train decreases:

$$50 - 20 \times 2 = 10$$

The number of blue wagons: $(50 - 20 \times 2) \div 2 = 5$.

Students look back the solution by checking their answer: $15 \times 2 + 5 \times 4 = 50$ cm.

Analysis of Consolidation Task. A train with the length of 100 cm including 36 wagons, how many red wagons and blue wagons are there?

The aim of this task is to help students consolidate what they have studied. They use their procedure to solve this problem by using temporary assumption.

The number of blue wagons: $(100 - 36 \times 2) \div 2 = 14$.

Analysis of Quiz. There are 33 liters of fish sauce contained in 2-litter bottles and 5-liter bottles. The number of bottles used is 12. Find the number of 2-liter bottles and 5-liter bottles used. Given that, all of bottles are full of fish sauce.

This is an application task. Students can solve this task as homework. Students learn how to apply what they have studied from the lesson to solve a realistic problem. Students recognise that the difference between one 5-liter bottle and one 2-litter bottle is 3 liters.

The number of 5-liter bottles: $(33 - 12\times2) \div 3 = 3$. Thus, the answer is 9 two-liter bottles and 3 five-liter bottles.

ANALYSIS OF THE VIDEOTAPED LESSON

The lesson is videotaped and analyzed using the video recording and the transcript. The actual lesson included several activities. The analysis in this section will be conducted by dividing the actual lesson into three stages: introductory activities, activities for task 1, and activities for task 2 and task 3. Each stage will be described and analyzed.

Introductory activities

1. Students were asked to make a train of 5 wagons by using 2 cm-red cards and 4 cm-blue cards.
2. Students were asked to use only 5 cards to make their own trains.
3. Students discussed in small groups of 4 students to list as many abilities as possible.
4. Students had to recognize the relationship between the length of the train and the numbers of red wagons, blue wagons.
5. Students had to recognize the restricted condition for each specific case in finding two numbers that their sum is known.
6. From the established table students had to understand that: if the number of red wagons increases one, then the length of the train decreases 2 cm.

In the lesson, some students made only one train of 5 wagons as required and then stop working.

Teacher asked students to paste their answer on blackboard. Most of the answers were presented except the two last options: 5 red wagons and 0 blue wagon, or 0 red wagon and 5 blue wagons

Teacher asked students to arrange the data following a systematic list.

S: There are many answers to this task.

T: Can you check your answer?

S: My train has 5 wagons including 2 red and 3 blue wagons. The length of the train is: $2 \times 2 + 3 \times 4 = 16$ cm.

T: We call "*the length of the train*" the restricted condition. Can you identify another restricted condition?

S: 14 cm.

T: How many red wagons and blue wagons in this train?

S: 3 and 2. We have $3 \times 2 + 2 \times 4 = 14$ cm.

Activities for Task 1

1. Students were asked to make a train with the length of 16 cm by using 2 cm-red cards and 4 cm-blue cards.
2. Students were asked to use some cards to make their own trains with the same length of 16 cm.

3. Students discussed in small group of 4 students to list as many abilities as possible.
4. Students had to recognize the relationship between the fixed length of the train and the numbers of red wagons, blue wagons.
5. Students had to know to analyse a natural number into sum of two natural numbers with specific restricted conditions.

In this task, students received an almost blank table; some cells have numbers that helped students fill the data into the table easier.

Students analysed number 16 as follows:

$16 = 8 \times 2 + 0 \times 4$

$16 = 6 \times 2 + 1 \times 4$

$16 = 4 \times 2 + 2 \times 4$

$16 = 2 \times 2 + 3 \times 4$

$16 = 0 \times 2 + 4 \times 4$

T: What is given?

S: The length of the train is 16 cm.

T: If the train has 6 wagons, how many red wagons and blue wagons in this train?

S: From the table I saw that this train has 4 red wagons and 2 blue wagons.

T: If we do not make the table, can you explain your solution?

S: If all 6 wagons are red, the train's length decreases 4 cm. So I got 2 blue wagons.

T: Who can express the answer by using mathematical operations?

S: $(16 - 6 \times 2) \div 2 = 4 \div 2 = 2$ (blue wagons).

Activities for Task 2 and Task 3

1. Students were asked to solve an extended problem that is difficult to guess and check.
2. Students were required to create a procedure to solve the task with a specific restricted condition.
3. Students were asked to present their answer by using mathematical operations?

In this task, some students used mental calculations or "*guess and check*" strategy to find out the answers. But they could not explain the answer logically.

S: There are 15 red wagons: $15 \times 2 = 30$ cm. And 5 blue wagons: $5 \times 4 = 20$ cm.

T: I ask you to give a procedure to solve this task not only use your mental calculation.

The teacher guided students to create a procedure by using the temporary assumption to solve the problem.

T: If 20 wagons are red, what is the length of the train?

S: 40 cm.

T: Why does the length decrease?

S: Because we replaced blue wagons by red wagons?

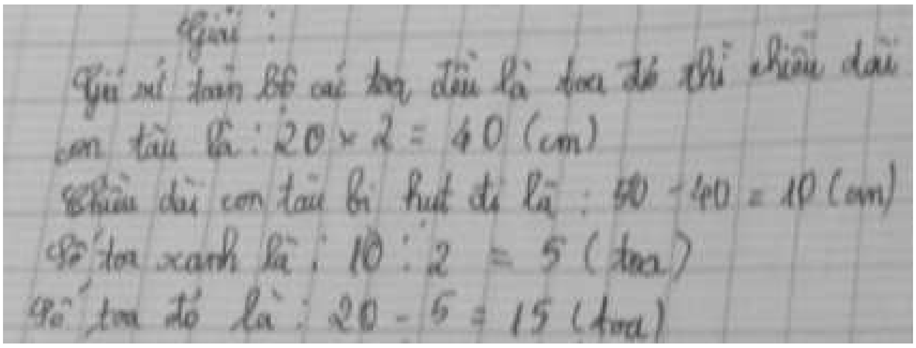
T: How many blue wagons did we replace?

S: 5 blue wagons.

T: How did you get 5?

S: (50 - 40) ÷ 2 = 5.

A student's solution:


Giải:
Giả sử toàn bộ các toa đều là toa đỏ thì chiều dài con tàu là: 20 × 2 = 40 (cm)
Chiều dài con tàu bị hụt đi là: 50 - 40 = 10 (cm)
Số toa xanh là: 10 : 2 = 5 (toa)
Số toa đỏ là: 20 - 5 = 15 (toa)

Translation into English of a student's solution:

If all wagons are red then the train's length is: 20 × 2 = 40 (cm).

The train's length decreases: 50 - 40 = 10 (cm).

The number of blue wagons: 10 ÷ 2 = 5 (wagons).

The number of red wagons: 20 - 5 = 15 (wagons).

The students applied the procedure to solve Task 3.

The number of blue wagons: (100 - 36 × 2) ÷ 2 = 14 (wagons).

The number of red wagons: 36 - 14 = 22 (wagons).

DISCUSSION AND CONCLUSION

Teaching primary school mathematics aims to equip young pupils with basic mathematics skills and develop their mathematical thinking to solve problems. Some senior classroom teachers have experienced to foster and develop students' mathematical thinking without theoretical background. Most of teachers in Vietnam really need a practical framework to develop pupils' mathematical thinking in their actual classrooms.

This lesson was prepared by a senior teacher, he has involved in some educational projects at primary level. As I analyzed the activities in the lesson by using videotaped recording, the teacher followed four main activities in a lesson that were suggested by the MoET to develop mathematical thinking.

In introductory activities, the task is a open-ended task, it helped students get start to observe many abilities, predict the length of the train by "guess and check". Teacher managed students to work and achieved the following aims:

- Examine the students' previous knowledge in finding the answer for:

  $\square \times 2 + \bigcirc \times 4 = ?$

  where $\square + \bigcirc = 5$.
- Consolidate the previous knowledge involved with new lesson: Find two numbers that their sum is 5. The new lesson needs to have one more restricted condition is the length of the train.

Find □ and ○ such that: □ × 2 + ○× 4 = 14 and □ + ○ = 5.

These activities gave students opportunities to show the ability of observing, predicting, rational reasoning and logical reasoning in solving problems related to the analysis of a natural number into the sum of two other numbers with a restricted condition.

In activities for task 1, teacher facilitated students explore mathematical knowledge and construct new knowledge by themselves. Students recognized the relationship between the fixed length of the train and the numbers of red wagons, blue wagons.

Find □ and ○, where □ × 2 + ○× 4 = 16, but □ + ○ = unknown. Students observed and predicted answers. Students created a procedure to solve the problem when □ + ○ = a fixed number. From specific situations students suggested a procedure to solve general problem. Students invented their own procedures or algorithms for solving problems.

In activities for task 2 and task 3, students practiced the new knowledge by solving exercises and problems given by teacher.

Students applied the analysis of natural number into sum of two other numbers with a restricted condition to solve some mathematics problems systematically by using temporary assumption. These two tasks examined the thinking operations that occurred in the lesson such as: comparison, generalization, and specialization.

Teacher concluded what students have learnt from new lesson and assigned the homework.

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Appendix

Mathematics Lesson Plan

Grade 5 (10-11 years old)

Teacher: Senior Teacher Mr. Tran Quang Khen, Le Qui Don Primary School, Hue City, Vietnam.

1. **Title**: Find two numbers that their sum and a restricted condition between them are known.
2. **About the research theme**
   - Nurturing ability of observing, predicting, rational reasoning and logical reasoning in solving problems related to the analysis of a natural number into the sum of two other numbers with a restricted condition.
   - Examining instruction that focuses on "applying the analysis of natural number into sum of two other numbers with a restricted condition to solve some mathematics problems systematically by using *temporary assumption*".
   - Examining the thinking operations that occur in the lesson such as: comparison, generalization and specialization.

In the national standard mathematics curriculum (2006) for primary level, we emphasize more in word problems that considered being good situations for pupils to explore and solve mathematical problems. Students' mathematical thinking will be enhanced when they solve word problems. Most of these problems are rooted from the real life situations.

At the beginning of Grade 4, students know how to solve and express solutions of problems having three operations of natural numbers.

Example. A toy train has 3 wagons with the length of 2 cm, and 2 wagons with the length of 4 cm. Find the length of the train.

Answer. $3 \times 2 + 2 \times 4 = 14$ (cm).

But if we set the problem in a reverse way:

> A toy train has two types of wagon: 2 cm- wagons and 4 cm - wagons. This train has the length of 14 cm including 5 wagons. Find the numbers of 2 cm- wagons and 4 cm - wagons of the train.
>
> The sum of two numbers needed to find is 5.
>
> Restricted condition: The total length of 2 cm-wagons and 4 cm-wagons is 14 cm.

This reverse problem is quite different with what students have learnt before. The problem requires them to analyze a natural number into sum of two other numbers logically.

3. **Goal**
   - For students to be able to recognize a number as a sum of two other number with a restricted condition;
   - Know how to find two numbers that their sum and a restricted condition are known.

4. Instruction plan

- Understanding the relationship of two related quantities;
- Identifying the restricted condition of the relationship of two quantities.

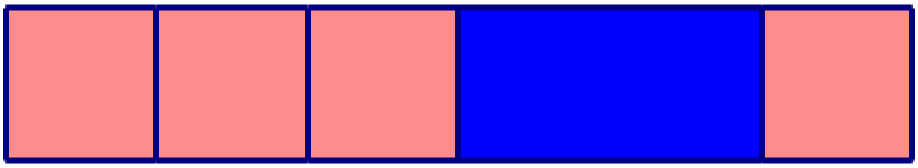
5. Instruction of the lesson

(1) Goal

- For students to realize that making a systematic list will help them understand the problem intuitively;
- Look for a pattern or procedure to solve a set of mathematical problems by using temporary assumption;
- Generalize the procedures obtained to solve some realistic problems;

(2) Flow of the lesson

Teacher prepares some red cards of 2 cm × 2 cm, and some blue card of 4 cm × 2 cm. Teacher called these cards to be wagons of a toy train.



Students are to use cards to make their own train.

Instructional Activities	Points for Consideration
Introductory Task. Use 2 cm cards and 4 cm cards to make a toy train of 5 wagons? List all abilities. How many answers can we get for this problem?	This is an open-ended task that requires students to make a systematic list of all abilities. Guess and check to get many answers.

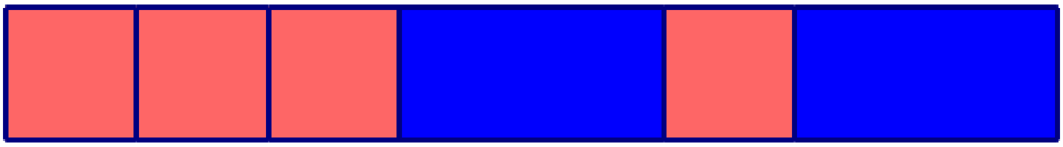
Students fill the data into the following table.

N. of red wagons		1		3		
N. of blue wagons	5		3	2		0
The length of the train in cm	20					

From the table of data, teacher supports students with following guided questions.

Question 0.1: If the number of red wagons increases one, then what is about the length of the train?	Recognize the relationship between the length and the numbers of red wagons, blue wagons. Recognize the difference between blue wagon and red wagon is 2 cm.
Question 0.2. A train with the length of 14 cm including 5 wagons. How many red wagons and blue wagons are there?	Understand with a restricted condition the answer will be unique.
Question 0.3. When the length of the train is	Identify the restricted condition for

longest? Shortest?	each case.
Task 1. Make a train with the length of 16 cm. List all the abilities and find the number of wagons in your trains?	Know how to analyze a natural number into sum of two natural numbers. Recognize the number of red wagons is an even number. Express the relationship between two quantities: If the number of blue wagons increases one the number of red wagons decreases two. This is also an open-ended task that requires students to make a systematic list of all abilities.



Students fill the data into the following table.

N. of red wagons	8			2	
N. of blue wagons		1			4
Total of wagons			6		

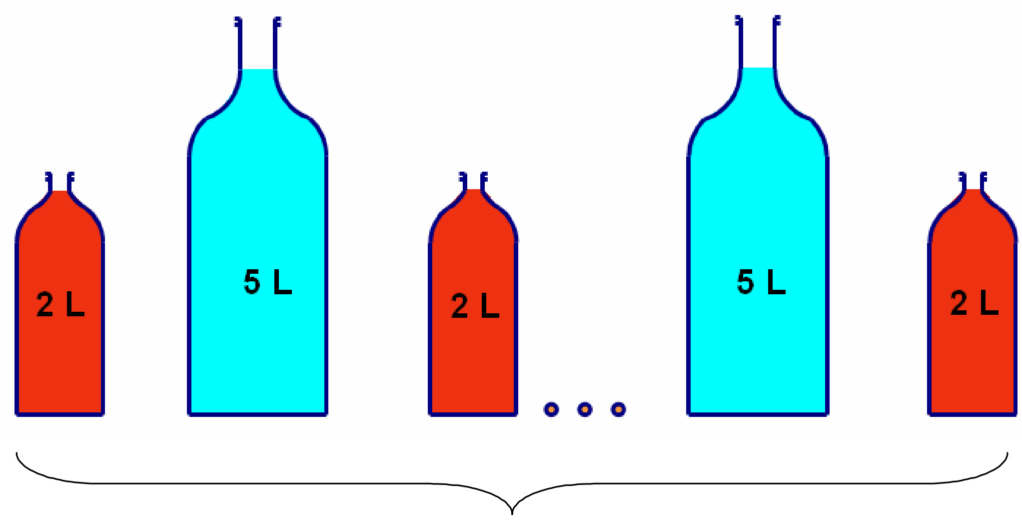
From the table of data, teacher supports students with following guided questions.

Question 1.1: A train with the length of 16 cm including 6 wagons. How many red wagons and blue wagons are there?	Predict the pattern to solve the general problems.
Question 1.2: When does the train have largest number of wagons? Smallest number of wagons?	Identify the restricted condition for each case.

Teacher encourage students create their own procedure to solve the problem.

Task 2. A train with the length of 50 cm including 20 wagons. How many red wagons and blue wagons are there?	The number of blue wagons: $(50 - 20\times2) \div 2 = 5$ Thus, the answer is 15 red and 5 blue wagons.
Question 2.1. If the train has only red wagons, what is the shortened length of the train? *Question* 2.2. In this case, you do not make a table. How can you find the number of blue wagons?	Create a procedure to solve the general problem.
Task 3 (Consolidation Task). A train with the length of 100 cm including 36 wagons, how	The number of blue wagons:

many red wagons and blue wagons are there?	(100 - 36×2) ÷ 2 = 14 Thus, the answer is 22 red and 14 blue wagons. Students practice the procedure just created to solve this problem.
Quiz. There are 33 liters of fish sauce contained in 2-liter bottles and 5-liter bottles. The number of bottles used is 12. Find the number of 2-liter bottles and 5-liter bottles used. Known that all bottles are full of fish sauce.	The number of 5-liter bottles: (33 - 12×2) ÷ 3 = 3. Thus, the answer is 9 two-liter bottles and 3 five-liter bottles. Students apply what they have learnt from the lesson to solve a realistic problem. Asking students to solve another problem to evaluate what the students are learning.



12 bottles containing 33 liters