Grades 7-12 Learning Progressions in Mathematics Content

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In grades K-6, we teach primarily arithmetic and simple geometry for one basic reason: as part of basic literacy. At the secondary level, we teach mathematics, or, perhaps more accurately, the *mathematical sciences* (including statistics, computer science, operations research, et al.) still as part of basic literacy but for several other major reasons: to be a wise consumer; to be an informed citizen capable of understanding issues of the day; to apply on the job; and (for only a few) to make discoveries that will expand the field. Because of its importance, throughout the world mathematics enjoys a status in schools second only to reading and writing in one's native language.

The mathematics curriculum has many sizes. From smallest to largest, they are:

the problem or episode	a few seconds to many minutes
the lesson	a class period or two
the chapter or unit	a few weeks
the course	typically, a half year or year
the school mathematics curriculum	K-12
the entire school curriculum	K-12 (all subjects)

The phrase "learning progressions in mathematics content" suggests "big ideas" that are at the size of the course or the school mathematics curriculum. That is, for the most part, these ideas take many months or several years to develop. These are the ideas that I discuss in this paper. But the good curriculum and the good teacher make many smaller progressions, often within an individual lesson, sometimes within a chapter or unit, and sometimes over an entire year.

Although this paper is concerned mainly with the specific grade range 7-12 (i.e., what is often termed *secondary education*), some of the learning progressions described here should typically begin in primary education earlier than grade 7, while others will go past grade 12 into tertiary education. Many of the progressions described here have been applied in developing the materials for grades 6-12 of the University of Chicago School Mathematics Project, but a casual look at the materials will usually not uncover the progressions because they tend to be beneath the surface.

My list contains nine progressions, not ordered by any measure of importance. It could have contained a few more. The list purposely ignores the sequencing of algorithms and the questions of the order of deduction in geometry. Algorithmic and logical sequences have formed the basis for virtually all school mathematics instruction over the years and are so familiar to mathematics educators that their repetition is not needed. However, I believe they should be seen in a particular perspective relative to the nine learning progressions, so I comment on them towards the end of this paper. Also, some progressions of slightly lesser importance have been omitted for lack of time and space.

Progression 1: from whole number to rational number to real number, and then to complex number and vector

I begin here because this progression is the earliest in schooling, beginning even before school. The key question here is: *What are the numerical objects of mathematics*?

The numerical objects obviously begin with the counting numbers, including 0. Early in the primary grades, through measurement and money, students should see that the counting numbers do not suffice and that negative numbers are natural in situations that have two opposite directions, such as above and below sea level, or profit and loss. By grade 7, we hope that the student realizes that symbols such as 116.42 and $\frac{5}{12}$ represent single numbers, not two numbers 5 and 12 with a mysterious bar between them, and that students also see 5 and +5 as the same, and -5 as a single number and not as a number with a mysterious sign.

The move from fractions and decimals to the rational numbers requires that students understand that the rational numbers are dense, that is, that we can find rational numbers as close to a given rational number as we like, both greater than it and less than it. The number line is a powerful representation for this move as it also is for the indication that a single number may be represented in a variety of ways.

These ideas are necessary for the progression to irrational numbers and real numbers, and for the idea of continuity that students will encounter in their study of functions. There are several ways to get from rationals to irrationals. One common way is via nested intervals. For $\sqrt{2}$, we square rational numbers to see if their squares are less than or greater than 2.

$1^2 < 2$ and $2 < 2^2$, so $1 < \sqrt{2} < 2$.

$1.4^2 < 2$ and $2 < 1.5^2$, so $1.4 < \sqrt{2} < 1.5$.

$1.41^2 < 2$ and $2 < 1.42^2$, so $1.41 < \sqrt{2} < 1.42$.

…and so on. And we conclude that $\sqrt{2}$ is described by a decimal that begins 1.41…

A problem with this sequence is that students come to think that real numbers *are* decimals rather than that they *can be represented by* decimals. So it is important to get at some real numbers directly. We can do it easily with $\sqrt{2}$ by noting that this is the length of a diagonal of a unit square. I have colleagues who have trouble with this idea; they think that lengths are really rational numbers because in everyday life we compute with rational numbers and not irrationals. I argue that since the the length of the diagonal of a real square is as close to $\sqrt{2}$ as the length of its side is to

1. A similar argument can be made for π and many other irrationals. We can then move to the notion that any infinite decimal represents a number.

This progression can branch from real numbers in three ways. A first branch is to vector. If numerical objects can represent points on a number line, then why not points in the plane? Addition and subtraction of vectors and the cross product for multiplication help students to appreciate the properties of operations of numbers with which they are familiar. And if objects can represent points on a line or points in a plane, then why not points in space? In school mathematics, we do not go beyond 3-dimensional vectors, but this progression continues into the study of linear algebra in college.

A second branch is to complex number. Although it is common and natural to introduce complex numbers as solutions to polynomial equations that cannot be solved in the reals, a disadvantage of this order is that students too easily interpret the terms *real* and *imaginary* as descriptors chosen because the set of numbers they describe *exist* and *do not exist*.

The concreteness of the complex numbers can come to play earlier if these numbers are associated with the coordinate plane as reals are with the number line, and their addition and multiplication are seen geometrically to generalize addition and multiplication of reals. One of the most interesting aspects of this connection between arithmetic and geometry is the fact that addition of complex numbers is easy in rectangular coordinates (that is, $(a, b) + (c, d) = (a + c, b + d)$), while multiplication is easy in polar coordinates ($[r, \theta] \bullet [s, \phi] = [rs, \theta + \phi]$), and turns DeMoivre's Theorem into a corollary.

Throughout this particular path of this progression from whole number to complex number, a student shouold view the arithmetic operations as being able to be interpreted both as binary operations (e.g., adding two numbers yields a third number) and as unary operations (adding by a particular constant number has its own properties). E.g., it is as unary operations that students learn what it means to add 0 to a number, and that adding n and subtracting n are inverse operations.

A third branch from real number is to the matrix as an object that can represent a single point, a finite set of points, a vector, or more generally, multi-dimensional data. Many students do not understand the properties of the operations of arithmetic until they have seen objects such as matrices for which important operations of addition and multiplication can be defined but do not possess all the field properties. The connection of matrices with vectors, which can wait until the tertiary level, brings the first and third branches together.

Progression 2: from numerical expression to algebraic expression, and then to function as a relationship and then to function as an object

The progression from numerical expression to algebraic expression includes with it the progression from *number* to *variable*.

In the primary school, the student should be introduced to two uses of the idea of *variable*: (1) variable as *unknown*, as in 3 + ___ = 10 or 3 + x = 10; (2) variable as *generalized arithmetic*, as in

A = LW as describing (area of a rectangle = length times width), or

$a + 0 = a$ as generalizing the instances (9.6 + 0 = 9.6) and ($\frac{2}{3} + 0 = \frac{2}{3}$).

In the secondary school, the student then can be introduced to a third important use: (3) variable as *function argument* or *parameter*, as in f: x $\rightarrow$ 3x + 5, in which the idea of a variable continuously varying first appears. At the tertiary level, a fourth use, (4) variable as *arbitrary symbol*, as in descriptions of the 4-group {I, *a*, *b*, *ab*} by the equations $a^2 = b^2 = I$; $ab = ba$ together with group properties.

A key idea in the progression from counting number to rational number is the treatment of a fraction as a single number. This idea, that a pair or larger group of symbols can be viewed as one, is called *chunking* by psychologists; it is the cognitive mechanism by which we view a string of letters as a single word, the cognitive mechanism underlying all of reading, and it is exceedingly important in the progression from arithmetic expression to algebraic expression.

Let us consider a common pattern used in textbooks, such as the length of a train in which the engine at the front is 30 meters and each car is 20 meters long. We ask for the length of the train. At the primary school level, students make a table.

Number of cars	Length of train
1	30 + 20 = 50
2	30 + 20 + 20 = 70
3	30 + 20 + 20 + 20 = 90
…	…

The next step in the progression is to convert the repeated additions to multiplication.

Number of cars	Length of train	Length of train
1	30 + 20 = 50	30 + 20 = 50
2	30 + 20 + 20 = 70	30 + 2•20 = 70
3	30 + 20 + 20 + 20 = 90	30 + 3•20 = 90
…	…	…

It is critical that the student understand the importance of the expressions 30 + 20, 30 + 2•20, and 30 + 3•20. They are not just for calculating the answer. They are for generating a pattern that will enable a person to find quickly the length of the train regardless of how many cars there are. A slight change in the table helps.

Number of cars	Length of train	Length of train
1	30 + 20 = 50	30 + **1**•20 = 50
2	30 + 20 + 20 = 70	30 + **2**•20 = 70
3	30 + 20 + 20 + 20 = 90	30 + **3**•20 = 90
…	…	…
n		30 + **n**•20

We now have an expression for the length of the train, using the variable to generalize the arithmetic. The expression represents a single number, the length, but also tells us how that length was calculated. We graph the ordered pairs (n, 20n + 30) as dots and find that the dots lie on a line. We have pictured a function. At this time, this function is a relationship – given the input *n*, the expression indicates the output 30 + 20*n*. When we write f(n) = 30 + 20n, we reinforce that idea. We are now using the variable *n* as an argument in a function. Only when the student sees and graphs many other relationships, noting that some are linear and some are not, does it make sense to try to categorize functions into linear, quadratic, etc. And then, when we look at the properties of these functions, it makes sense that we can name a function by a single letter. If we have used f(n) notation, the letter to use is naturally f.

This progression, from numerical relationship between pairs of numbers to thinking of the relationship as a single object, can take years, stretching from early to late secondary school and often into the tertiary level of mathematics study. It is helped by having operations on functions, such as function addition or function composition. The move to thinking of functions as objects requires that we have properties of classes of functions that are not the same as properties of individual functions. For instance, to assert that the set of linear functions is closed under composition requires that a student think of a function as an object. In my opinion, the practice of some mathematicians and in some technology that f(x) stands for a function hurts this progression. To me it is important to distinguish between the value of a function and the function itself.

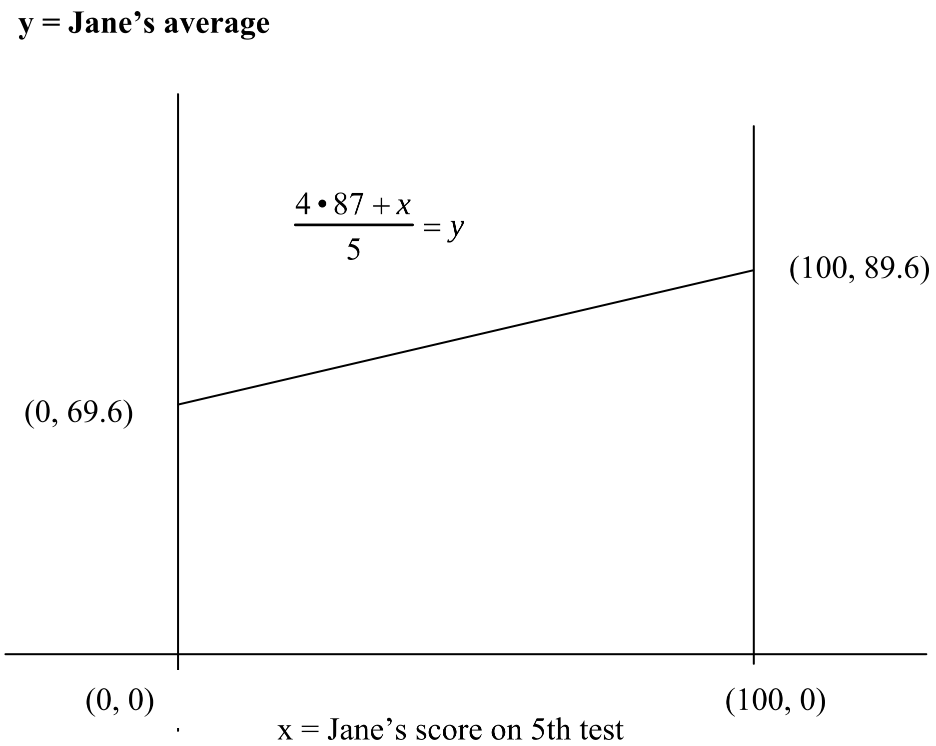
Here is another illustrative example that begins with a typical problem and shows how the progression is often poorly made.

Jane has an average of 87 after 4 tests. What score does she need on the 5th test to average 90 for all five tests?

When this question is given along with the study of algebra, the student is expected to let a variable such as x stand for Jane's score on the 5th test and to solve an equation such as $\frac{4 \bullet 87 + x}{5} = 90$. Here the variable is an unknown. But most students (and I have found, most teachers – even those with substantial mathematical knowledge) use arithmetic to solve the problem. This exposes a fundamental difficulty. Since the problem can be so easily solved without algebra, students naturally wonder why algebra is needed to find the unknown. Thus, though one reason for presenting this problem in a class is to show the power of algebra, the effect is the opposite. *Certain common problems that are supposed to help the progression from arithmetic to algebra actually hinder it.*

If we stop with just the solution to this problem, then we have shown that algebra is not needed, but most teachers do stop once they have the answer. To justify the use of algebra, we can generalize the problem. In the statement of the problem, replace 90 by y. (This is an easy step for us but certainly not for all students unless they have had some instruction.) If Jane's average for all 5 tests is y, then $\frac{4 \bullet 87 + x}{5} = y$. This is an equation for a linear function with slope $\frac{1}{5}$ and y-intercept $\frac{4 \bullet 87}{5}$ or 69.6. It shows that any point

Jane gets on any test contributes $\frac{1}{5}$ to her average. The graph of this equation for $0 \leq x \leq 100$, is a segment from (0, 69.6) to (100, 89.6), shows all the possible solutions.



By generalizing the pattern, we have now seen the power of algebra to solve an entire set of problems at once, something that arithmetic cannot do. And we have also changed the idea of x being an unknown to x and y being pattern generalizers and finally to x being an argument of a function.

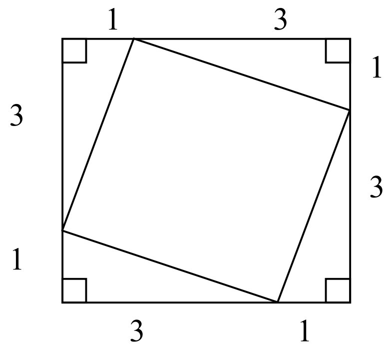
Progression 3: from properties of individual figures to general properties of all figures in a particular class

I use the phrase "class of figures" because the most obvious examples are geometric, but could just have easily used "set of objects". Breaking a set of objects into various subsets based on properties is a very important idea in mathematics; we classify numbers, functions, 2-dimensional geometric figures, 3-dimensional figures, matrices, transformations, etc. Our reason for doing so is because we want to deal with properties that are held by all objects in a set. And we want to do that because of efficiency.

For instance, we might have students solve individual quadratic equations by completing the square. But, if we have completed the square for the general case $ax^2 + bx + c = 0$ in order to develop the Quadratic Formula, there is no need to complete the square to solve any future quadratic equation. We might have students determine the length of the hypotenuse of a right triangle with legs 1 and 3 using an area argument such as in the following diagram, but if we do it in general, we have the Pythagorean Theorem (or the Theorem of Three Squares, as it is called in some of the economies represented here) and there is no need to use the area argument again.

Area of middle (tilted square) = area of large square – 4(area of right triangle)
= $16 - 4 \bullet 0.5 \bullet 1 \bullet 3$
= 10

So the side of square has length $\sqrt{10}$.



The generalization from individual instances to a general formula is a hallmark of mathematics and one of the progressions that students need to see again and again. If we have deduced that $\sqrt{2}$ is irrational, how many other numbers can we prove to be irrational by an analogous argument?

The move from properties of individual figures to properties of all figures in a set can be subtle and unsettling for students. For instance, consider the teacher who wishes to convince her students that the sum of the measures of the angles of a triangle is 180°. But what does this mean? There are subtleties here. Consider statements (1) and (2). They have much the same sentence structure. But there is quite a difference between them.

(1) In ΔABC, $AB + BC + AC = 15$.

(2) In ΔABC, $m\angle A + m\angle B + m\angle C = 180°$.

Statement (1) applies only to certain triangles and is given information in some problems, while statement (2) applies to all triangles and is often isolated as a theorem.

Here is another example of the same type where the statements look even more alike. Each of these statements could be true.

(3) In a triangle, the largest angle is obtuse.

(4) In a triangle, the smallest angle is acute.

Here, statement (3) could be given information about a particular triangle while statement (4) is true for all triangles.

Young children are aware of properties of individual triangles, such as in (1) or (3). But the study of geometry requires that students be able to work with properties of all triangles, such as (2) and (4). These semantic similarities between statements get in the way, and they motivate the use of quantifiers.

(1) It is sometimes true that in ΔABC, $AB + BC + AC = 15$.

(2) It is always true that in ΔABC, $m\angle A + m\angle B + m\angle C = 180°$.

(3) In some triangles, the largest angle is obtuse.

(4) In all triangles, the smallest angle is acute.

How do we know that (2) is always true? Typically, a good teacher gives students an activity: draw a triangle on a sheet of paper, carefully measure its angles, and add the measures.

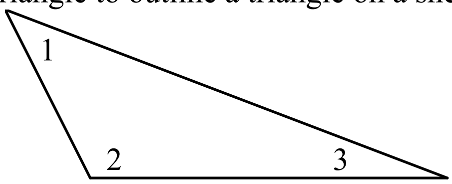
What happens? Though most of the sums students obtain from measuring are near 180°, they are not all exactly 180°. The teacher may explain that measurements are not exact, but some students wonder whether statement (2) is really true always. Maybe the teacher is oversimplifying, just as is done with spelling rules in English such as "i before e except after c", being that there are exceptions such as height and weight, not all of them weird. Maybe the sums of angle measures round to 180°. Maybe the sum is 180° only for triangles within a certain range of shapes. Maybe the average sum is 180°.

There is now a quandary regarding how to proceed because one of the points the teacher wants to make is that you *cannot* make a generalization for infinitely many objects just by looking at specific examples. The activity seems like a perfect hands-on activity but it has failed to help the progression from individual figures to a class of figures. The difficulty is that the strategy used by Miss Smith is fine for asserting the truth of $m\angle A + m\angle B + m\angle C = 180°$ for a *particular* ΔABC, and even for *many particular* triangles but not for *all* triangles.

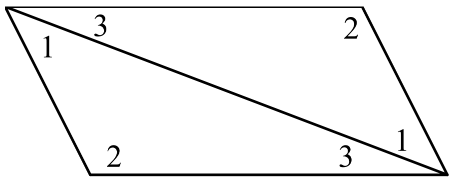
Here is a better activity for making the progression.

Step 1: Cut out a triangle from a sheet of cardboard.

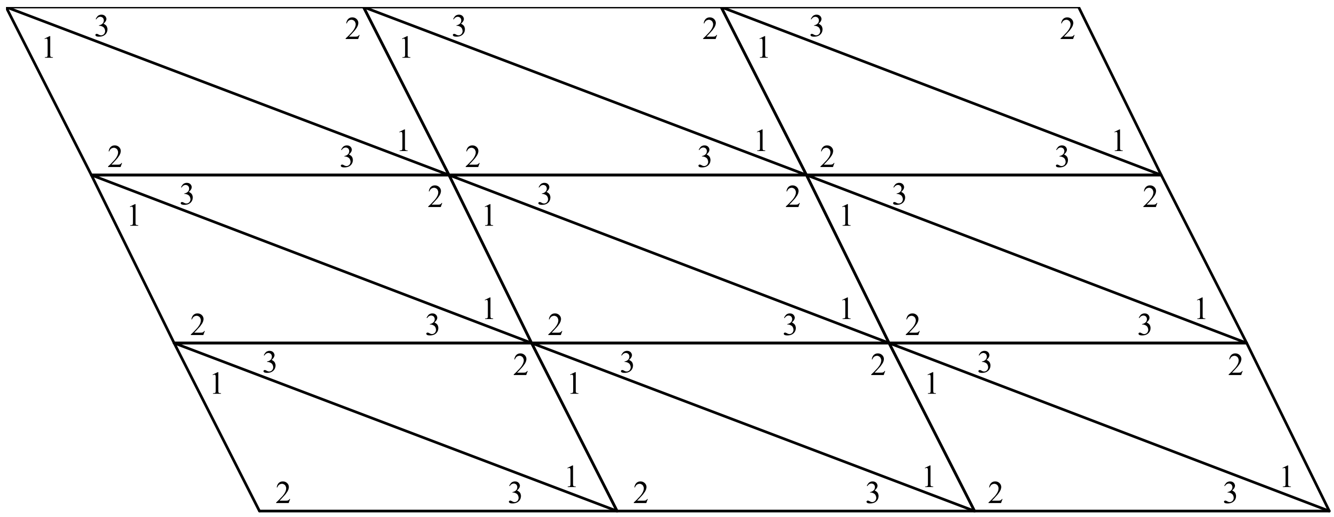
Step 2. Use this triangle to outline a triangle on a sheet of paper.



Step 3. Outline the same triangle again and turn the second triangle so that the two triangles together form a parallelogram.



Step 3: Repeat the parallelogram to tessellate the plane.



When the parallelograms are tessellated, we see that around each vertex are six angles, two copies of each angle of the original triangle. Since the sum of the six angles is 360°, the sum of three different angles is 180°. Now we see that the sum is 180° because it is half of the number 360° that is used for a complete revolution. And we should tell why the Babylonians chose 360°. And we might even tell our students that angle measure does not have to be in degrees. If another unit were used, then the sum would be different. And students should know why this tessellation could not be done on the Earth's surface, not because they would necessarily have that on a test, but to see the significance of parallel lines in this argument and just in case someone asked.

Here a dynamic geometry drawing program such as *Cabri Geometrie* or *Geometer's Sketchpad* can help, because it enables the student to verify large numbers of examples with triangles of all different shapes. But the technology does not suffice; Paul Goldenberg has reported that many students think the computer is pre-programmed to get the results it shows. Deduction is needed.

Progression 4: from inductive arguments to deductive ones and then to deduction within a mathematical system

In the preceding progressions, I have used inductive thinking several times. Inductive thinking is how we live most of our lives. We walk or ride to work using a route that in the past has served us well. We try out something in the classroom and, if it works, we'll use it again and again. Induction is also one of the two mechanisms by which we reason in mathematics. Induction gives us conjectures.

Induction is often misused in school. Children are asked: What is the next number in this sequence? 1, 2, 3, 4, 5, 6, 7, 8, 9, … The correct answer: There is not enough information to tell. It could be 10 if the sequence is that of the counting numbers in increasing order. It could be 11, if the sequence is that of the positive integers not divisible by 10. It could be 0 if the sequence is the sequence of the units digits of the positive integers.

Induction can be subtle. Consider 0^0. It seems reasonable to view 0^0 as the limit of 0^x as the real number x approaches 0. We calculate: $0^2 = 0$; $0^1 = 0$; $0^{0.5} = \sqrt{0} = 0$; $0^{0.1} = \sqrt[10]{0} = 0$; and so on. It seems definitive: $0^0 = 0$.

But it also seems reasonable to view 0^0 as the limit of x^0 as x approaches 0. Now we calculate: $2^0 = 1$; $1^0 = 1$; $(0.5)^0 = 1$; $(0.1)^0 = 1$. It seems just as definitive: $0^0 = 1$. We tried to induce the value of 0^0 and came up with two different values depending on which pattern we wished to follow. So a first step in the progression is to show that induction does not always work even in a closed mathematical setting.

Deduction begins with a single word "if" or "suppose" or "assume", followed by a question "What if?" or "Then what happens?" *Assumptions are important in deduction*. What if the sequence 1, 2, 3, 4, 5, 6, 7, 8, 9, … is the sequence of integers in increasing order that are the days of the months of the year beginning with January? Then we have 1, 2, 3, …, 31, 1, 2, 3, …, 28 (or 29, depending on the year), … It's not the simple sequence we thought!

Deduction is the hallmark of mathematical thinking. We have not taught students the essence of mathematical thought unless they appreciate the power of deduction. The full power, however, only comes when we are aware of the assumptions from which we deduce. Those assumptions are perhaps easier to see in applications, where assumptions become constraints in a problem, than in theory, where assumptions often need to be traced back to a large number of postulates.

You have saved 500 baht. What if you save an additional 150 baht each week? Then what happens? This open-ended question is the essence of mathematical thinking. Too often we tell students what we want them to prove rather than asking them to prove *anything* from the given information and then see how far they can go.

We can do this also with pure mathematics. Divisibilty properties are very suitable for early deduction and many students are curious about them. Suppose m and n are any different integers, each divisible by 7. What can be deduced about m + n? (Some students begin by thinking that m + n is always divisible by 14; deduction shows that m + n is always divisible by 7 and a counterexample shows that m + n is not always divisible by 14.) What can be deduced about mn? What can be deduced about $m^3n^2 + m^2n^3$? Students can discover as well as deduce properties of divisibility.

Deduction often carries us into mathematics at a higher level. The human population of our planet is currently about 6.8 billion and growing at a rate of 1.17% a year according to recent estimates. If we assume those estimates are correct and the growth rate is constant, then what? A first reasonable conclusion is that the population P of the planet is given by $P = 6{,}800{,}000{,}000(1.0117)^n$, where n is the number of years from now.

A nice aspect of this problem is that the question of the domain of n becomes significant to the application and is more than a textbook exercise. Can n be 0? (Yes, that is the population "now".) Can *n* be negative? (Yes, for example, if n = -2, then the formula calculates the population *two years ago* under the assumptions of the problem.) Must *n* be an integer? (No, but then we are forced into asking when exactly "now" is.) This example can be used to give meaning to non-integer and negative exponents.

In this population example, when $n = 1000$, $P \approx 766{,}000{,}000{,}000{,}000$, that is, about 766 trillion people, or over 5,000,000 people per square kilometer of land. In contrast, Macau, the most dense country in the world, has a density of under 19,000 people per square kilometer. What seems to be a weakness of the formula we have used is actually its strength, because we can change the assumptions and so deduce a new formula. We might want to indicate a limit L for the world population; if so, we can obtain a logistic formula for the world population *n* years from now, describable by the recursive formula $P(n+1) = P(n) + .0117P(n)\left(1-\frac{P(n)}{L}\right)$. We may have moved into tertiary mathematics, but we have also given secondary school students a reason for studying that mathematics. In this way, applied mathematics can give at least as much meaning to deduction as pure mathematics.

To understand deduction fully, students also need to see examples where false assumptions lead to nonsense. Perhaps my favorite example of this type is to ask students for any two integers between 0 and 100. Say that the integers are 48 and 61. Then ask for two more, say 9 and 91. Now I assert that I will prove: If 48 = 61, then 9 = 91. One possible proof is as follows:

$$\begin{aligned}
& & 48 &= 61 \\
&\Rightarrow & 48 \bullet \frac{1}{13} &= 61 \bullet \frac{1}{13} \\
&\Rightarrow & \frac{48}{13} &= \frac{61}{13} \\
&\Rightarrow & 3\frac{9}{13} &= 4\frac{9}{13} \\
&\Rightarrow & 3 &= 4 \\
&\Rightarrow & 3 \bullet 82 &= 4 \bullet 82 \\
&\Rightarrow & 246 &= 328 \\
&\Rightarrow & 246 - 237 &= 328 - 237 \\
&\Rightarrow & 9 &= 91
\end{aligned}$$

The point is that even if a person uses valid reasoning, if that reasoning stems from statements that are not true, then you cannot be certain of the truth of any conclusion. But if you use valid reasoning from true statements, then you must get true statements. And, if you reason from a given statement whose truth you do not know, but you get a false statement, then you know that the given statement had to be false. This, of course, is the foundation of indirect proof.

Somewhere before they leave their study of mathematics, students need to be introduced to the wonders of a mathematical system, that is, a system where deduction from a small number of postulates has, over the years, led mathematicians not only already to deduce a myriad of theorems but to be continually proving more theorems. Either Euclidean geometry, or the real number complete ordered field, or a set of postulates for the positive integers are good candidates for such a system. They are good candidates because there are an unlimited number of theorems in each system and so they display the immense power of deduction.

Progression 5: from uses of numbers to uses of operations to modelling with functions

I am a passionate believer that students should see the wonders of pure mathematics, but I am at least as passionate in believing that students must be introduced to the breadth of applications of our subject. If a student leaves a mathematics classroom not knowing why the mathematics is important, it is our fault. We cannot expect teachers of other subjects to tell students why they need to study mathematics. That is our job.

Modeling begins in the primary school. In primary school, we expect that students have seen numbers used as *counts* and as *measures*, with counting units and units of measure, respectively. As mentioned earlier, they should also have seen that, in situations with two directions, positive and negative numbers arise. Fractions and percent show that numbers are also the result of *ratio comparisons*, and such numbers do not have units. π is a wonderful example of an irrational number used as a ratio comparison Numbers also represent *locations*. Street addresses, rank orders, and scales such as the Centigrade scale for temperatures or the decibel scale for sound intensity represent this fourth use of numbers. Numbers also may be used as *identification or codes*, as in charge card numbers or ISBNs. And of course there are numbers simply used as numbers, such as when we examine prime numbers or lucky numbers.

From the uses of numbers develop meanings for the operations. The sum x + y has meaning if x and y are counts or measures, but not necessarily when x and y are ratio comparisons, and almost never if x and y are codes. When x and y are locations such as scale values, the sum x + y does not have much meaning – the sum of two temperatures does not have meaning, for example, yet the difference x – y almost always has a meaning. Each of the operations has fundamental meanings: addition as putting-together or slide; subtraction as take-away or comparison, and special cases of comparison are error and change. Multiplication is area or acting across, size change, or rate factor; division is rate or ratio; powering is growth. The meanings are related to each other just as the operations are: for examples, take-away undoes putting-together; size change undoes ratio.

As a progression, it is fundamentally important that in the primary school these uses of numbers and operations go beyond counts to include non-integers. Then, in the lower secondary school, these uses can be employed to give meaning to algebraic expressions. For instance, in the expression $\frac{y_2 - y_1}{x_2 - x_1}$ for the rate of change between (x_1, y_1) and (x_2, y_2), $y_2 - y_1$ and $x_2 - x_1$ are subtraction comparisons and the division is a rate, so it is no surprise that $\frac{y_2 - y_1}{x_2 - x_1}$ represents rate of change.

From the meanings of algebraic expressions come the situations that functions model. When items with unit costs x, y, and z are purchased in quantities A, B, and C, the sum Ax + By + Cz is an addition putting together rate factor multiplications to arrive at a total price. Linear functions arise from these linear combinations or situations of constant increase or constant decrease. Exponential functions model situations of growth or decay. Quadratic functions model situations of acceleration or deceleration (the rate of a rate), or area. Trigonometric functions model circular motion and are often quite appropriate in situations where phenomena occur periodically. The broad kinds of situations that the various types of functions model should be as much a part of the curriculum as the mathematical properties of these functions, for it is almost certain we would not be studying them were it not for their applications.

It is useful at this point to consider the three levels of modeling: the *exact model*, such as in the number of games necessary for *n* teams to play each other; the *almost-exact theory-based* model, such as in modeling the path of a thrown ball by taking measurements along its path; and the *impressionistic* model, such as when one finds that the population of a region over a particular time interval is described well by a quadratic function and no theory explains that. We sometimes give the incorrect impression that mathematical models are always approximate and messy, but the reality is that many mathematical models are exact. On the other hand, some users of mathematics give the alternate impression – that their impressionistic models are reliable – and we need to caution against that improper inference.

Progression 6: from estimation of a single measurement to statistics for sets of numbers, and from descriptive statistics to inferential statistics

In many quarters, probability and statistics are considered together and separate from other mathematics. I see these two topics for the most part as instances of other progressions. For instance, the calculation of relative frequency is an example of ratio division. The fitting of lines or curves to data is how we model data by functions. Still, there is a progression that is distinctively statistical, namely the consideration of data sets (rather than a single data set) to make inferences about situations of variability, and the role of randomness.

The public often has the view that mathematics is an exact science, and that estimates are never as good as exact values. How wrong this view is! There are often times that estimates are to be preferred over exact values. Consider (1) predictions such as the lifetime of a light bulb or the lifetime of an individual person or the price of chicken next year, or (2) values that are changing constantly such as temperatures or populations, or (3) measurements such as a person's waist size or score on a memory test, or (4) values that we want to be consistent in a table, such as 3-place decimals in describing the winning percentages of sports teams. In these cases, estimates by convenient nearby numbers, intervals, or distributions are more appropriate models than an exact value. For convenience we may substitute a single number or two to describe an interval or a distribution, numbers we call statistics.

Thus this progression begins with the importance of estimates. Then it moves to consideration of how to describe (estimate) a set of numbers without listing all the numbers. We may use single numbers such as the mean (an example of rate division) or the median (a location), or pairs of numbers as with an interval, or multiple locations such as the 5-number summary (minimum, 1st quartile, median, 2nd quartile, maximum) – five numbers used as locations! We realize that we have lost information in the use of these statistics, so we often return to the full distribution and describe it with such terms as skewness, symmetry, tails, its modes and outliers, and its spread, with statistics such as the standard deviation or mean absolute deviation.

It is often said that statistics is different from mathematics because statistical thinking is probabilistic and inferential, while mathematical thinking is deductive. True, but good statistics uses deduction from hypotheses just as mathematics does. The major difference, in my opinion, is that statistics is applied mathematics in that it arises from data, while mathematics arises from theory. To make this distinction, it is better to use the term *relative frequency distribution* for a distribution based on data, and *probability distribution* for one based on theory, rather than the terms *experimental probability* and *theoretical probability* found in many places. The better terms emphasize that probabilities are always either assumed (often through randomness or from past experience) or calculated from assumed probabilities, whereas relative requencies always arise from data. For instance, in Malaysia in 2007 provisional figures from the U.N. indicate that 235,359 males and 221,084 females were born. Thus the relative frequency of male births was about 0.516. Assuming randomness, the probability that a randomly-selected baby born in Malaysia in 2007 is a boy is , so by making the randomness assumption we can turn the relative frequency into a probability, but far more likely for calculations a person would use 0.516 or 0.52 for the probability.

It is useful to have distributions that arise from data that are not random (such as test scores) and data that are randomly generated from experiments (such as coin-tossing), because we often want to pick a data point at random from a non-random distribution (for instance, if we choose a student at random from a non-random distribution, what is the probability that the test score is greater than some number). This is preparation for the idea of events with low probability, that is, events that are not very likely to happen.

Although it is not uncommon to separate statistics from other mathematics, there are several advantages to teaching them together. First, statistics requires dealing with expressions involving absolute value, square roots, binomials, and other algebraic language. Second, distributions are functions and can be used to strengthen function concepts such as end behavior, symmetry, and limits. Third, distributions can be modeled by functions, such as when linear regression is used to determine the line of best fit for a set of data. Fourth, transformations that are applied to data such as scaling and translating in order to normalize the data are often also applied to functions in order to study their behavior.

Once students have dealt with data and they, students are ready for hypothesis testing and inferential statistics. For instance, one might ask about the Malaysian 2007 births, could a ratio this far from 50% males and 50% females have occurred by chance? In other words, *if* (i.e., *hypothesizing that*) the sex of a baby is random between males and females, what is the probability that 235,359 males would be born out of 456,443 births. In this way, null hypotheses and alternate hypotheses are assumptions made for a particular situation and thus give another opportunity for deduction. The only difference is that the answers to questions of inference are typically probabilities ("The probability is x that data like these would arise if the data were random."). In this case, the probability is very, very small that such numbers of males and females would occur. With small numbers, we would calculate a probability like this using binomial coefficients; with large numbers such as these, we use a normal distribution. But calculating is not the only way. Students should learn simulation such as the use of Monte Carlo techniques. Among all the direct statistical tests, I think it is easiest to begin hypothesis testing with Chi-square tests. Finally, a major goal of teaching this content should be to immerse students in examples of how statistics can be used to gain valuable information about and make inferences from data in order to combat the common societal view that statistics are not to be trusted.

Progression 7: from the idea of same size and shape (same shape) to a general definition of congruence (and similarity) applying to all figures, to conditions for the congruence (and similarity) of simple geometric figures, to the application to all figures and graphs and the Graph Transformation Theorems

The treatments of congruence and similarity in K-12 schooling do not typically follow a smooth path. Congruence in lower grades is “same size, same shape”, applying to all figures, yet when a concerted study is begun in later grades, the figures are often restricted to be triangles and perhaps circles. Later, perhaps only in college, a definition of congruence in terms of transformations is provided that brings the student back to consideration of all figures. In my opinion, this is not the best progression. The restriction of congruence to simple figures is not at all helpful for student understanding of the idea.

Throughout schooling it is possible to consider figures as congruent if and only if one can be mapped onto the other by a composite of reflections, rotations, translations, and glide reflections (or any one of many other equivalent definitions). This gives an intuitive picture that can be reinforced by reference to products produced by a machine, tessellations, duplicate copies of a photograph, etc.

There are two advantages to this sequence. By considering the graph of a (typical elementary) function as a set of points in the plane, the idea of congruence easily extends to the congruence of graphs. A particular special case are graphs that are translation images of each other. Equations for these graphs can be found using the Graph Translation Theorem: In a relation described by a sentence in x and y, the following two processes yield the same graph:
(1) replacing x by $x–h$ and y by $y-k$; and
(2) applying the translation $T(x,y) = (x+h, y+k)$ to the graph of the original relation.
It is somewhat surprising that this theorem is not found in many of today’s books, given the number of corollaries that are found. Call the original relation the parent and its translation images the offspring of the relation. Then the following correspond:

Shape of graph	**Parent**	**Offspring (image)**
Line	$y = mx$	$y – y_0 = m(x – x_0)$ Poinr-Slope
Line	$y = mx$	$y – b = mx$ Slope-Intercept
Circle	$x^2 + y^2 = r^2$	$(x – h)^2 + (y – k)^2 = r^2$
Parabola	$y = ax^2$	$y – k = a(x – h)^2$
Sine Wave	$y = \sin x$	$y = \sin(x – c)$ Phase Shift

Parabola intercepts	$ax^2 = c \Leftrightarrow x = \pm\sqrt{\frac{c}{a}}$	$a(x-h)^2 = c \Leftrightarrow x = h\pm\sqrt{\frac{c}{a}}$ Quadratic Formula
Exponential	$y = b^x$	$y = ab^x$
Logarithmic	$y = \log_b(x)$	$y = \log_b(ax)$

Similarity is often restricted in K-12 schooling to polygons and polyhedra. This, too, is unfortunate, as so many interesting instances of similarity involve more complex figures. Every day students see pictures on television and other media that are similar to the actual objects being pictured. In the upper primary and lower secondary school, scale drawings, physical models of large objects, and maps can be used to demonstrate similarity. Dilatations (size changes) can be introduced to create larger and smaller images of given figures. A definition of similar figures in terms of transformations is easily given once there has been the corresponding definition for congruent figures. The Fundamental Theorem of Similarity, that in similar figures with ratio of similitude k, corresponding angles have equal measures, corresponding lengths are in the ratio k, corresponding areas are in the ratio k^2, and corresponding volumes are in the ratio k^3 can be applied to the question of the existence of giants such as those students have read about in fairy tales and see in cartoons. Under a general definition of similarity, all parabolas are similar and so are all graphs of exponential and logarithmic functions.

These properties of the graphs of functions follow from the Graph Scale Change Theorem: Translation Theorem: In a relation described by a sentence in x and y, the following two processes yield the same graph:

(1) replacing x by $\frac{x}{a}$ and y by $\frac{y}{b}$; and

(2) applying the scale change $T(x,y) = (ax, by)$ to the graph of the original relation.

As is the case with the Graph Translation Theorem, this is a powerful theorem with many useful corollaries that are important precursors for the study of integrals in calculus and assist in the understanding of graphs of all functions.

Shape of graph	**Parent**	**Offspring (image)**
Circle	$x^2 + y^2 = 1$	Ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$
Hyperbola	$xy = 1$	*Hyperbola: $xy = k$
Parabola	$y = x^2$	*Parabola: $y = ax^2$
Line	$y = x$	*Line: $y = mx$
Sine Wave	$y = \sin x$	Amplitude/Period: $y = A\sin(Bx)$
Exponential	$y = b^x$	*Change of Base: $y = ac^x$
Logarithmic	$y = \log_b(x)$	*Change o Basey = $\log_c(x)$

*Image is geometrically similar to the parent.

Progression 8: from scientific calculators to graphing calculators to computer algebra systems

In many economies the major role played by mathematics beyond arithmetic is – whether intentional or incidental – as a sorter; that is, to separate out better students from poorer ones regardless of their interests or vocational goals. In years past, using mathematics as a sorter was defensible because not many people would be helped by knowing mathematics beyond arithmetic and simple geometry and because the examination questions involved skills that were needed by the small percent of the population who needed higher mathematics. However, today we feel that most people would be helped by knowing about the behavior of functions, the fundamentals of descriptive and inferential statistics, and many other mathematical topics not found in the primary school curriculum. And today there exist hand-held devices and computer software that can accomplish any of the difficult calculations that often served to sort students in the past. These tools make it possible for the first time to realize the goal of significant mathematical literacy for all. A corollary to this argument is that if one does not allow this technology, then the sorting done by mathematics is often due to performance on tasks that can be carried out automatically by a machine and not on a person's ability to understand and apply mathematical ideas.

Most of the individuals charged with the task of creating standards in our economies today were in school when there were no hand-held calculators. These individuals, almost all of whom were very successful in their mathematics study, often dismiss today's calculator and computer technology not only as unnecessary but even as harmful to the mathematics education of today's youth. I profoundly disagree.

Since 1975 we at Chicago have been developing curricula using the latest technology – first, just scientific calculators, then graphing calculators, and most recently, computer algebra systems (CAS). Our experiences have convinced me and those who work with me that this technology enhances both the conceptual understanding and problem-solving ability of students. For slower students, those who in the past might have been sorted out of mathematics and even out of advanced schooling because of difficulties with our subject, the technology is particularly important. It enables them to understand mathematics that would otherwise defeat them. It is life support, helping them to survive their mathematics courses, or it is a crutch, helping them until they can walk. For better students, the technology is an *extender*, helping them to move more easily into more advanced mathematics. And for all students, the technology sends a message – that mathematics is current and relevant in today's world.

Today's advanced technology is so sophisticated that it, like any other advanced concept, can overwhelm students who have not had experience with the corresponding work that is not so advanced. The order is straightforward: from early primary school, students should be working with calculators. In later primary school, scientific calculators that can deal with fractions should be used. In early secondary school, students should begin working with calculators that enable graphing and geometry. And, in later secondary school, students should have calculators with computer algebra system capability. The CAS technology, the newest in the arsenal of technology that can do mathematics, we have found to be particularly important for students who have had trouble learning algebra. For the first time, they can play with algebraic patterns with confidence that what they are doing will lead to correct answers, and by seeing the patterns in the answers, they gain proficiency.

These practical reasons for a technology sequence have a theoretical counterpart in the well-known paper-and-pencil algorithms that traditionally dominate the learning of arithmetic and algebra. Paper-and-pencil is a technology whose applicability hundreds of years ago, when paper was scarce and pens required ink, was strikingly analogous to the situation today in that the more affluent people and societies have access while the poorer lag. Schools could not begin to teach everyone the paper-and-pencil algorithms until almost all students had their own paper and ink supplies, and it is still the case in some schools that these are scarce commodities and rationed.

The procedures employed to obtain answers in arithmetic and algebra are carefully sequenced in today's books. We can also expect the analogous procedures to obtain answers using calculators to be carefully sequenced; the only problem is that many new and more powerful calculators are getting on the market each year.

To those who believe that paper-and-pencil work *is* mathematics, and calculator work is not, we might note that large numbers of students in the United States blindly apply paper-and-pencil algorithms with no idea of why they work or whether their answers make sense. They cannot multiply $\frac{3}{4}$ by 8 unless they change the 8 to $\frac{8}{1}$. To multiply 357 by 8000, they first put down three rows of zeros (not just three zeros). They are totally at a loss to explain long division. Algebra is even less understood. Polynomials are factored with no idea that if a number is substituted for the variable, the value of the original and factored expression will be the same. Equations are solved with no idea why one would ever want to solve an equation. Rational expressions are operated on with no idea of how to check whether the answer is correct except to look in the back of the book and hope that an answer is there.

Having the technology does not automatically eliminate these deficiencies, but it enables both student and teacher to spend time on the important ideas and not lose the forest through the trees. In almost all situations, paper-and-pencil manipulation should be a means to an end, not the end itself.

But it is certainly the case that some students overuse calculators, just as many o us use paper and pencil to calculate answers that we should have memorized. In our experience, this is particularly true of students who did *not* have calculators while they were learning the algorithms. Such students see calculators merely as a time-saver and do not understand their use in helping to learn facts and algorithms and to check work. Students who have calculators while they are learning mental and paper-and-pencil arithmetic are forced from the start to make decisions about when it is appropriate to use any or all of these means and seem to be able to make wiser decision later about the use of any of these technologies.

Progression 9: from a view of mathematics as a set of memorized facts to seeing mathematics as interrelated ideas accessible through a variety of means.

The sequence of ideas in mathematics can proceed logically as it does in many economies with the teaching of geometry and in many college courses. It can proceed algorithmically, that is, by the complexity of the algorithms, as it has traditionally done in our teaching of arithmetic, algebra, and calculus. Some have tried courses in which the mathematics of a topic proceeds in the order of the historical development of the topic. In the previous eight progressions, the mathematics proceeds from the cognitively simple to the cognitively more complex. This is the vertical dimension of school mathematics, from bottom to top, from lower grades to higher grades.

But there has to be a horizontal dimension, in which the algorithmic order, the logical order, the mathematical modeling, and the representations all play roles in the student's learning of the mathematics being studied. To know *how* to do some mathematics without knowing *why* you would want to do it, or *why it works*, or how to know you are right is insufficient. Cognitive scientists tell us that being able to connect and categorize helps learning. They also view representation and metaphor as the ultimate tests of whether someone understands a particular idea. Students need to be able to check their work by appealing to logic, or an application, or a representation. In the UCSMP curriculum, we call this the SPUR approach to understanding – **S**kills, **P**roperties, **U**ses, and **R**epresentations.

Consider for example the concept of absolute value. **S**kills associated with this concept: calculating $|x|$ for any value of x; solving sentences such as $|x| = k$, $|x| > k$, $|x| < k$, $|x - a| = k$, $|ax + b| < c$, $|x^2 + bx| = 10x$, and so on, of increasing complexity.

Properties associated with absolute value include the definition:
$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$; $x \leq |x|$ for all x; $|xy| = |x||y|$ for all x and y; $|x| + |y| \geq |x + y|$, and so on.

Uses associated with absolute value come from the idea of *distance*, namely, $|x|$ is the distance from x to 0 on the number line; $|x – y|$ is the distance from x to y . Special cases of distance are (undirected) change; error; comparison. For instance, in manufacturing an object when the error between the desired length L of the object and the actual length is A, the error is $|L – A|$, and if lengths are measured in millimeters and we wish that error to be less than 0.1 mm, then the object's length must satisfy $|x – A| < 0.1$

Representations of absolute value are on the number line or coordinate plane. or instance, the solutions to the inequality $|x – A| < 0.1$ are all points within 0.1 of A on a number line, or we can graph $y = |x – A|$ in the coordinate plane and look for the values of x corresponding to those values of y that are within 0.1 of the x-axis.

The SPUR dimensions of understanding of a general concept can be applied to specific situations as well and can illustrate interrelationships among the first eight progressions. Consider this cartoon that appeared in a daily newspaper in the United States.

The numbers 9.9 and 9.2 here are statistics; in fact, they are means. To us the cartoon is humorous, finishing with a common joke when a mean of counts is not a whole number. But to many people, the 9.9 and 9.2 indicate the lack of reality of mathematics. Just as you cannot have a "point-two" friend, you cannot trust statistics. And if students have not made the transition from whole number to rational number, and if they have not dealt with statistics, they will have great difficulty getting this joke.

The understanding that seems to be lacking here is the notion that in the process of gaining simplicity by using a single number to describe a set of numbers, something is always lost. Here we have single numbers describing entire distributions, and we have lost the distributions.

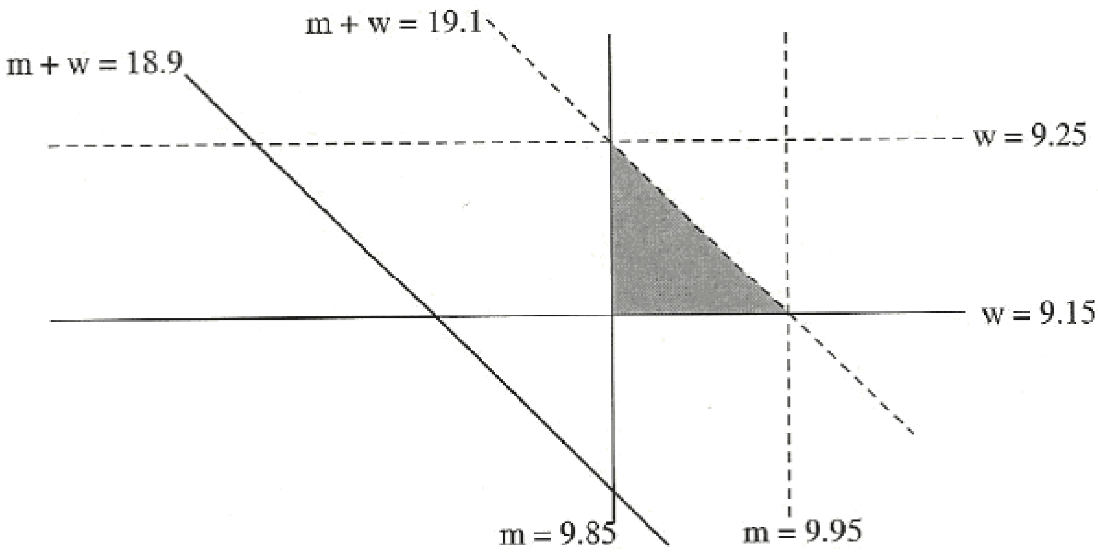
The next day the cartoonist continued this theme.

When I saw this cartoon, I became intrigued. Assuming the same number of men and women participated in the poll, how can you get 9.5 as an average of 9.9 and 9.2? Thus begins a mathematical analysis of the situation.

The numbers 9.9, 9.2, and 9.5 are all rounded to the nearest tenth. Each number stands for an interval. If m and w are the values for the average number of close friends a man and a woman have, then $\mathbf{9.85 \le m < 9.95}$ and $\mathbf{9.15 \le w < 9.25}$. (I am assuming we are rounding up all decimals that end in 5.) Within these intervals we wish to know whether it is possible to have $9.45 \le \frac{m+w}{2} < 9.55$, or, equivalently, $\mathbf{18.9 \le m + w < 19.1}$.

So one way to answer the question is to give pairs of values of m and w that satisfy the three inequalities written above in bold.

To find some pairs is not particularly difficult, but it seems like a very difficult problem to find all possible values. But if we examine the graphical representation, algebra, geometry, and statistics come together in a beautiful way. The graph of $9.85 \leq m < 9.95$ is a vertical stripe; the graph of $9.15 \leq w < 9.25$ is a horizontal stripe, and the graph of
$18.9 \leq m + w < 19.1$ is an oblique stripe between the lines $m + w = 18.9$ and $m + w = 19.1$. And all stripes contain their lower boundaries but not their upper boundaries, as shown on the next page.



The values of m and w that satisfy all three inequalities provide ordered pairs (m,w) that are either on or in a triangle. For instance, one pair of values is 9.88 for m, 9.17 for w, and so m+w = 19.05. So it could have been that the men in the study had, on average, 9.88 friends and the women 9.17 friends. They would have an average of 9.525 friends. And when these numbers are rounded, we get 9.9 friends for men, 9.2 friends for women, and 9.5 for the entire group just as we wanted.

This example involves algebraic skills (the solving of a system of inequalities), properties (realizing the meaning of a measurement to a single decimal place as well as the principles underlying the transformation of the inequalities into nice form), uses (the modeling of friendship by a rational number), and representations (the graph and the geometric representation of the algebra).

This example is cute but was not picked because it is cute. The horizontal integration of mathematics is as important as its vertical progression; otherwise students naturally believe that if they can answer questions involving dozens of separate mathematical ideas, then they have learned mathematics well. But they have not learned one of the most messages of mathematics: that the mathematics they are studying operates within a single logical system that ranges from everyday arithmetic through the most complicated of functions that one studies in analysis and includes measurement, algebra, geometry, trigonometry, probability, and statistics along the way.

It is possible to extend each of the progressions described here to the mathematics that students may encounter in college. Linear and exponential models pave the way for the consideration of logistic models. The transformations basic to Euclidean geometry lay the foundation for affine and non-linear transformations. Relationships within and between algebra and geometry exemplify the morphisms found in higher algebra. And it has been impossible to cover all the bases of mathematics in grades 7-12. The study of combinatorics, probability, and limits, and the logic of definition and of propositions came to mind as I wrote this essay and I am certain there are other important topics I have missed. I have also not dealt with nurturing the affective dimension of schooling – that, whatever we do, we have not succeeded with an individual student unless that student views mathematics without fear, with the desire to learn more, and with the awe that our subject deserves.

Learning Progressions and Standards

The essay above is an expansion of the paper I wrote for the APEC conference, a paper whose title was given to me. At the conference, the paper was presented in the session dealing with "standards" and I was asked by the organizers also to add some comments in this regard.

We recognize that standards can play a variety of roles in mathematics education, both in curriculum and evaluation. (1) Standards may determine curriculum, forcing all materials in a particular economy or geographic area to adhere to them. (2) Standards may guide curriculum, serving as suggestions to which an ideal curriculum might aspire. (3) Standards may represent criteria for minimal performance in order to move to a higher level. (4) Standards may set goals for high performance at a given level. Typically, evaluation standards are more explicit than curriculum standards, though sometimes the same standards are used to determine both curriculum and evaluation.

It is not uncommon to see standards conceptualized as a two-dimensional matrix, in which one dimension consists of strands or areas of mathematics and a second dimension is grade levels. Thought of in that way, five progressions in this paper lie as follows:

Progression 1:	Number
Progression 2:	Algebra
Progression 3:	Measurement
Progression 6:	Statistics
Progression 7:	Geometry

Three other progressions are in the realms of process standards, i.e., they cross all content areas. They are related to three of the four SPUR dimensions of understanding described in Progression 9.

Progression 4:	Reasoning (Properties)
Progression 5:	Modeling (Uses)
Progression 8:	Algorithmic Thinking (Skills)

The progressions mentioned at the start of this paper as being so common they not need be explicated here are those of deduction and of algorithms, and are related to Progressions 4 and 8 within this paper. The horizontal Progression 9 can be viewed as an integrative progression tying together the other eight areas and in which the fourth SPUR dimension, Representations, plays a major role.

Although the learning progressions described in this paper are, for the most part, directed at the secondary (grades 7-12) level, I have purposely not tried to be more specific and identify a particular year or years for a particular aspect of a progression. Although it is often useful, both in theory and in practice, to treat students of the same age as if they are cognitively alike, they do differ, and some are ready for particular ideas earlier than others. This readiness depends to a great extent on the expectations set in earlier years both in the home and in school, on the time that a student has in which to devote to mathematics, on the interest that the student has or displays in learning mathematics, and, in some economies, on the ability of the home to provide support for student learning. These criteria for readiness are more significant at the secondary level than at the primary level and they account for the fact that in many economies, the mathematical requirements for students begin to be differentiated at the secondary level on the basis of student performance and/or interest. Virtually all economies have realized that, at the secondary level, a “one size fits all” set of standards is not workable and, at some time, there needs to be what mathematics is for all students and what mathematics is for those who express more interest, desire, or ability.

There are very few observations that can be made about the learning of mathematics that apply in all cultures, but one of them, known from the very first international studies of the 1960s, is that *time-on-task* is a significant variable in performance. In the high-performing economies on international comparisons of mathematics performance (Singapore; Korea; Japan and Hong Kong, China), students put in large amounts of time outside the classroom, often in organized tutoring centers (sometimes called “tuition” or “juku”) or with individual tutors. These same conditions, not as well-organized, exist in high-performance public and private schools in the United States. In economies where vast numbers of students do not attend secondary school, or must work or do chores many hours a week in addition to their schooling, or have little or no access to technology, students cannot be expected to proceed through these learning progressions as quickly.

Further arguing against identifying particular grade levels for aspects of a particular learning progression is the ever-changing world of mathematics itself. The learning progressions related to the paper-and-pencil solving of equations in algebra are challenged by technology that can solve the most difficult of these equations in the same way that it solves the easiest. Statistics, which a half-century ago was rarely mentioned in standards, is now viewed in many economies as an important area of secondary mathematics, taking time that used to be devoted to other mathematical areas. As recent as twenty-five years ago, mathematical modeling was viewed in most economies as a tertiary area. Dynamic geometry technology has changed the ways in which we view geometrical objects.

The lack of specificity in the progressions is also due to my view that there are innumerable ways to approach mathematics meaningfully. In this paper I have suggested some ways that I hope will provoke others to examine the mathematics represented in their standards and the materials used in their economies and organize learning progressions that are suitable for the students in their economies.